Chapter 2. **ALGEBRA**

This section deals with algebraic properties of the sets W_n and how they illustrate the fact of relativity of mathematics. We begin with the most basic algebraic equation $a \times_n x = b$. Now, due to the rules of arithmetic in any W_n we have the following cases. Suppose $a \in W_n$ such that $a^{-1} \notin W_n$, then any of the following can occur: we can have a unique solution, e.g. $3 \in W_2$, $3^{-1} \notin W_2$, and $x = 1$ is the unique solution of $3 \times_2 x = 3$; many solutions, e.g. $0.3 \in W_2$, $0.3^{-1} \notin W_2$, and $x = 0.1, 0.11, ..., 0.19$ are the solutions of $0.3 \times_2 x = 0.03$; and no solutions, e.g. $3 \in W_2$, $3^{-1} \notin W_2$, and there is no solution to $3 \times_2 x = 1$. The next case is when there is a unique inverse a^{-1} for $a \in W_n$, then we have the following fact: $a \times_n x = b$ either has a unique solution or no solutions. That the equation has many solutions does not occur here. To see this, first note, that a unique inverse cannot exist if $|a| < 1$. Now, write the equation as $a_0.a_1...a_n \times_n x_0.x_1...x_n = b$ with $a_0 \neq 0$ and assume a solution exists. Then if we vary x_n between 0 and 9 the $a_0 \cdot 0.\underbrace{0...0}_{n-1} x_n$ term of the product will also vary, thus changing the product and invalidating the equality, hence the solution must be unique. Finally, we consider the case where $\left|\left\{a^{-1}\right\}\right| > 1$. The following is then true: $a \times_n x = b$ has either many solutions or no solutions. To see this, write $a_0.a_1...a_n \times_n x_0.x_1...x_n = b$ and assume that there is a solution. Now, note that if we vary x_n between 0 and 9 the term $0.\underbrace{0...0}_{n-2} a_{n-1} \cdot 0.\underbrace{0...0}_{n-1} x_n$ of the product is irrelevant since, by definition, it drops off and we get many solutions.

Now we will show the independence of existence of solutions of the equation $a \times_n x = b$ by varying n. The cases that arise are as follows: if there exists a unique solution in W_n, that does not necessarily imply the existence of a solution in W_m for $m \neq n$. However, if there are many solutions to an equation in W_n, there will be the same number of solutions in W_m, but not necessarily the same ones. Here are some examples: $2 \times_2 x = 0.01$ has no solution, but $2 \times_4 x = 0.01$ has a unique solution $x = 0.005$. Both $3 \times_2 x = 18$ and $3 \times_4 x = 18$ have a unique solution $x = 6$. The equation $0.1 \times_2 x = 0.12$ has 10 solutions $\{1.2, 1.21, ..., 1.29\}$ and $0.1 \times_4 x = 0.12$ also has 10 solutions, $\{1.2, 1.2001, ..., 1.2009\}$. Note, that the solutions are different. Also, notice the two equations $0.1 \times_2 x = 0.12$ and $1 \times_2 x = 1.2 \Leftrightarrow x = 1.2$ are not equivalent due to different number of solutions.

We now consider systems of linear equations. Let us start with a special case. We know that in W_2, $2^{-1} = 0.5$ and $0.5^{-1} = \{2, 2.01, ..., 2.09\}$, then the system $\begin{cases} 2 \times_2 x = 0.32 \\ 2.01 \times_2 x = 0.32 \\ \quad ... \\ 2.09 \times_2 x = 0.32 \end{cases}$ has a unique solution $x = 0.16$, moreover each equation in the system also has $x = 0.16$ as a unique solution. In fact, we have the following theorem: the system $a_i \times_n x = b$ such that $a_i \in \{a^{-1}\}$ for some $a \in W_n$ either has no solution (in this case each equation has no solution) or has a unique solution (in this case, each equation has the same unique solution).

Next we consider systems of two linear equations with two unknowns, their solutions in W_n and W_m for $n \neq m$, and also show that systems can be nonequivalent after elementary row operations. For example, consider $\begin{cases} 0.14 \times_2 x +_2 0.23 \times_2 y = 0.22 \\ 0.61 \times_2 x +_2 0.43 \times_2 y = 0.76 \end{cases}$, then, for example, $x = 0.83$ and $y = 0.79$ is a solution, and therefore, there are actually 100 solutions in W_2: $\{(0.8, 0.7) \quad (0.8, 0.71) \quad ... \quad (0.8, 0.79)\}$, $\{(0.81, 0.7) \quad (0.81, 0.71) \quad ... \quad (0.81, 0.79)\}$, …, $\{(0.89, 0.7) \quad (0.89, 0.71) \quad ... \quad (0.89, 0.79)\}$. Now, consider $\begin{cases} 0.14 \times_4 x +_4 0.23 \times_4 y = 0.22 \\ 0.61 \times_4 x +_4 0.43 \times_4 y = 0.76 \end{cases}$, then an easy computation shows that any solution of the W_2 system is not a solution in W_4. To see this, take the minimal solution from W_2, then $0.14 \times_4 0.8 +_4 0.23 \times_4 0.7 = 0.273$ and obviously any other solution will produce a larger result, hence cannot be a solution of this system. Now, by computing the solution to the system (using regular real numbers), we get numbers that in W_2 are $x = 1$ and $y = 0.35$, then by incrementing these values by 0.01, we see that there can be no solutions in W_4.

On the other hand, consider $\begin{cases} 10 \times_4 x +_4 20 \times_4 y = 0.07 \\ 20 \times_4 x +_4 10 \times_4 y = 0.05 \end{cases}$. This system has a (in fact, unique) solution $x = 0.0010$, $y = 0.0030$, whereas the system $\begin{cases} 10 \times_2 x +_2 20 \times_2 y = 0.07 \\ 20 \times_2 x +_2 10 \times_2 y = 0.05 \end{cases}$ has no solution. Thus, the order of m and n has no influence on solutions. Other situations are also possible. For example, $\begin{cases} 1 \times_n x +_n 1 \times_n y = 3 \\ 2 \times_n x +_n 1 \times_n y = 4 \end{cases}$ has a solution ($x = 1$, $y = 2$) for $n = 2, 4$, whereas the system $\begin{cases} 1 \times_n x +_n 1 \times_n y = 3 \\ 2 \times_n x +_n 2 \times_n y = 5 \end{cases}$ has solutions for neither values of n.

Let us consider the problem of determining equivalency between systems and their elementary transformation (via row operations). Given $\begin{cases} 0.14\times_2 x +_2 0.23\times_2 y = 0.22 & (1) \\ 0.61\times_2 x +_2 0.43\times_2 y = 0.76 & (2) \end{cases}$, consider $\begin{cases} 0.14\times_2 x +_2 0.23\times_2 y = 0.22 & (1) \\ 0.75\times_2 x +_2 0.66\times_2 y = 0.98 & (1)+(2) \end{cases}$. Now, ignore the possibility of noncommutativity and pick any solution, e.g. $(0.8, 0.7)$, of the first system and plug it into the second system. An easy computation shows that the solution does not satisfy the $(1)+(2)$. In fact, no other solution will satisfy it, hence the two systems are nonequivalent. Next, consider $\begin{cases} 0.14\times_2 x +_2 0.23\times_2 y = 0.22 & (1) \\ 1.22\times_2 x +_2 0.86\times_2 y = 1.52 & (2)\cdot 2 \end{cases}$. Again, ignore the possibility of noncommutativity and pick a solution, e.g. $(0.81, 0.71)$, to the system with rows (1) and (2), then it easy to see that it does not satisfy the system with rows (1) and $(2)\cdot 2$. In fact, all other solutions except $(0.8, 0.7)$ do not satisfy this system, hence, again, the systems are not equivalent. Similar analysis shows that the system $\begin{cases} 0.14\times_2 x +_2 0.23\times_2 y = 0.22 & (1) \\ 6.24\times_2 x +_2 4.53\times_2 y = 7.82 & (1)+(2)\cdot 10 \end{cases}$ is not equivalent to the original system.

Therefore, the elementary row operations produce nonequivalent systems of equations.

Here is another example. Consider the following system: $\begin{cases} 1\times_n x +_n 1\times_n y = 1 \\ 0.11\times_n x +_n 0.37\times_n y = 0.44 \end{cases}$.

Now, no matter that $\begin{vmatrix} 1 & 1 \\ 0.11 & 0.37 \end{vmatrix} \neq 0$ for any $n \geq 2$, we have, for example, that there are solutions for $n = 3, 5, 6, 7, 9$, and yet no solutions for $n = 2, 4, 8$.

We move now to the Cartesian product $\underbrace{V_n \times ... \times V_n}_{k}$. This is just the standard Cartesian product, with the natural addition and constant multiplication:

$\left(x_1,...,x_k\right) +_n \left(y_1,...,y_k\right) = \left(x_1 +_n y_1,...,x_k +_n y_k\right)$ and $\alpha \times_n \left(x_1,...,x_k\right) = \left(\alpha \times_n x_1,...,\alpha \times_n x_k\right)$ for $x_1,...,x_k, y_1,...,y_n, \alpha \in W_n$. Now, in order for this product to make sense to a W_n-observer, it must be such that $1 \le k \le \underbrace{9...9}_{n}$. We can work with the standard notions when $k = 2$ - plane, and $k = 3$ - space. The classical axioms of a linear space are also valid here whenever $x_1,...,x_k, y_1,...,y_n, \alpha \in W_{Ent[0.3n]}$, but in general, these properties are not valid due to lack of associativity and distributivity.

Now what is left is to define $\dim V_n$. We introduce two alternative definitions. We first define $\dim_1 V_n = \max s$, where s is the index of $u_0, u_1, ..., u_s$ such that $u_0 \in W_n$,

$u_1 \in W_n \setminus \left\{W_n \times_n u_0\right\}$ such that $\left\{W_n \times_n u_0\right\} \not\subset \left\{W_n \times_n u_1\right\}$;

$u_2 \in W_n \setminus \left(\left\{W_n \times_n u_0\right\} +_n \left\{W_n \times_n u_1\right\}\right)$ such that $\left\{W_n \times_n u_0\right\} \not\subset \left\{W_n \times_n u_2\right\}$ and $\left\{W_n \times_n u_1\right\} \not\subset \left\{W_n \times_n u_2\right\}$;

…

$u_k \in W_n \setminus \left(...\left(\left(\left\{W_n \times_n u_0\right\} +_n \left\{W_n \times_n u_1\right\}\right) +_n \left\{W_n \times_n u_1\right\}\right) +_n ... +_n \left\{W_n \times_n u_{k-1}\right\}\right)$ such that $\left\{W_n \times_n u_0\right\},...,\left\{W_n \times_n u_{k-1}\right\} \not\subset \left\{W_n \times_n u_k\right\}$ and finally,

$W_n \setminus \left(...\left(\left(\left\{W_n \times_n u_0\right\} +_n \left\{W_n \times_n u_1\right\}\right) +_n \left\{W_n \times_n u_1\right\}\right) +_n ... +_n \left\{W_n \times_n u_{s-1}\right\}\right) \neq \varnothing$, but

$W_n \setminus \left(...\left(\left(\left\{W_n \times_n u_0\right\} +_n \left\{W_n \times_n u_1\right\}\right) +_n \left\{W_n \times_n u_1\right\}\right) +_n ... +_n \left\{W_n \times_n u_s\right\}\right) = \varnothing$.

The second dimension, $\dim_2 V_n = \max s$ where s is the index of $u_0, u_1, ..., u_s$ such that $u_0, u_1, ..., u_s \in W_n$ and $\{W_n \times_n u_i\} \not\subset \{W_n \times_n u_j\}$ for $i < j$ and $i = 0, ..., s-1$, and $j = 1, ..., s$ and $W_n \setminus \left(...\left(\left(\{W_n \times_n u_0\} +_n \{W_n \times_n u_1\} \right) +_n \{W_n \times_n u_1\} \right) +_n ... +_n \{W_n \times_n u_{s-1}\} \right) \neq \varnothing$, but $W_n \setminus \left(...\left(\left(\{W_n \times_n u_0\} +_n \{W_n \times_n u_1\} \right) +_n \{W_n \times_n u_1\} \right) +_n ... +_n \{W_n \times_n u_s\} \right) = \varnothing$.

From the point of view of an observer with a higher level of thickness, we have the following theorem: $\dim_i \underbrace{V_n \times ... \times V_n}_{k} = \left(\dim_i V_n\right)^k$ for $i = 1, 2$. Now, the relationship between the two definitions can be expressed in the following theorem: $\dim_2 V_n \geq \dim_1 V_n$.

Here is a useful result when dealing with $W_2 : \dim_1 V_2 \geq 7$. To show equality, consider the set of elements $\{99.99, 99.98, 99.97, 99.95, 99.92, 99.90, 99.53\}$, we will show that this set spans W_2. Consider the following set

$$A = \{V_2 \times_2 99.99\} \cap \{V_2 \times_2 99.98\} \cap \{V_2 \times_2 99.97\} \cap \{V_2 \times_2 99.95\} \cap \{V_2 \times_2 99.92\} \cap$$
$$\cap \{V_2 \times_2 99.90\} \cap \{V_2 \times_2 99.53\}$$

Now, this set has 199 points, moreover $\{V_2 \times_2 99.99\} \setminus A = \{\pm 99.99\}$, $\{V_2 \times_2 99.98\} \setminus A = \{\pm 99.98\}$, $\{V_2 \times_2 99.97\} \setminus A = \{\pm 99.97\}$, $\{V_2 \times_2 99.95\} \setminus A = \{\pm 99.95\}$, $\{V_2 \times_2 99.92\} \setminus A = \{\pm 99.92\}$, $\{V_2 \times_2 99.90\} \setminus A = \{\pm 99.90\}$ and $\{V_2 \times_2 99.53\} \setminus A = \{\pm 99.53\}$ - See Appendices 1-7. Finally, to see that

$$W_2 = \left(\left(\left(\left(\left(\{V_2 \times_2 99.99\} +_2 \{V_2 \times_2 99.98\}\right) +_2 \{V_2 \times_2 99.97\}\right) +_2 \{V_2 \times_2 99.95\}\right) +_2 \{V_2 \times_2 99.92\}\right) +_2 \{V_2 \times_2 99.90\}\right)$$

- see Appendices 8-13.

We can also have the following cases occur: $\{V_2 \times_2 98.99\} \cap \{V_2 \times_2 99.01\} = \{0\}$ so that we have two lines contained in W_2 intersecting only at zero. Also, we have the following theorem $W_2 = \left(\left(\{V_2 \times_2 99.01\} +_2 \{V_2 \times_2 98.99\}\right) +_2 \{V_2 \times_2 95.51\}\right)$, moreover these three lines intersect only at zero.

Now we can consider the plane $V_2 \times_2 98.99 +_2 V_2 \times_2 0.01$ that lies entirely on the line $V_2 \times_2 1$. Note, that $V_2 \times_2 0.01 = \{0, \pm 0.01, ..., \pm 0.99\}$ and we can show that $V_2 \times_2 98.99 +_2 V_2 \times_2 0.01$ actual equals W_2, i.e. this plane coincides with the line.

Also we have that $V_2 \times_2 98.99 \cap V_2 \times_2 99.01 = \{0\}$, i.e. the space $V_2 \times_2 98.99 +_2 V_2 \times_2 99.01$ is generated by two intersecting (only at zero) systems of collinear vectors. Now, take $98.03 \in V_2 \times_2 98.99 +_2 V_2 \times_2 99.01 = B$ and consider $V_2 \times_2 98.03 \cap B$. Also $|V_2 \times_2 98.03 \cap B| = 31$ and hence $W_2 = \left(\left(V_2 \times_2 98.99 +_2 V_2 \times_2 99.01\right) +_2 V_2 \times_2 98.03\right)$.