Chapter 3. GEOMETRY

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In Chapter 2 we have defined a $k-$fold Cartesian product of W_n's. In particular, $W_n \times W_n$ is a plane and $W_n \times W_n \times W_n$ is a space. Now, a line in the plane will be defined to be $\{(x,y) \in W_n \times W_n \mid a \times_n x +_n b \times_n x +_n c = 0 \text{ for some } a,b,c \in W_n\}$ and a plane in a space will be defined to be $\{(x,y,z) \in W_n \times W_n \times W_n \mid a \times_n x +_n b \times_n x +_n c \times_n x +_n d = 0 \text{ for some } a,b,c,d \in W_n\}$. In particular, for $u \in W_n$ we have $\{W_n \times_n u\}$ - a line on a plane and simultaneously, when viewed as $y = x \times_n u$, a line on a line $\{W_n \times_n 1\}$. Also, for $u_1, u_2 \in W_n$ we have the plane $\{W_n \times_n u_1\} +_n \{W_n \times_n u_2\}$ that lies in space, but when viewed as $u_1 \times_2 x + u_2 \times_2 y = z$, it is a plane on the line $\{W_n \times_n 1\}$. In fact, we can also have a space $\{W_n \times_n u_1\} +_n \{W_n \times_n u_2\} +_n \{W_n \times_n u_3\}$ on that same line, etc. Also, as stated in Chapter 2, we can have a line containing two lines that intersect only at zero.

Now, let us consider intersection of lines on the plane $W_2 \times W_2$. Here are a few examples of how two lines that intersect in the usual sense actually intersect at no, one, two, ten and even a hundred points. For no intersection, consider $\begin{cases} 0.08 \times_2 x +_2 0.78 \times_2 y +_2 0.09 = 0 \\ -0.47 \times_2 x -_2 0.75 \times_2 y -_2 0.38 = 0 \end{cases}$, for one point $\begin{cases} 0.59 \times_2 x +_2 0.79 \times_2 y +_2 0.59 = 0 \\ 1.00 \times_2 x +_2 1.00 \times_2 y +_2 0.41 = 0 \end{cases}$, for two points $\begin{cases} 0.31 \times_2 x +_2 1.00 \times_2 y +_2 0.63 = 0 \\ 1.00 \times_2 x +_2 0.34 \times_2 y +_2 0.91 = 0 \end{cases}$, for ten points $\begin{cases} 0.30 \times_2 x +_2 1.00 \times_2 y +_2 0.53 = 0 \\ 0.32 \times_2 x +_2 0.28 \times_2 y +_2 0.74 = 0 \end{cases}$, and, finally, for 100 points in the intersection, consider $\begin{cases} 0.14 \times_2 x +_2 0.23 \times_2 y -_2 0.22 = 0 \\ 0.61 \times_2 x +_2 0.43 \times_2 y -_2 0.76 = 0 \end{cases}$. For a visual illustration of each, see Appendices 14 - 18.

Now, we can also have different situations occur depending on the observers. We can have two given lines intersecting in a plane for any W_n, not intersecting for any W_n, or intersect for some

W_n, but not intersect for W_m such that $m \neq n$. In general, given 2-fold Cartesian products of $W_{n_1}, \ldots, W_{n_k}$ with $n_1 < \ldots < n_k$ there exists two lines with coefficients in W_{n_1}, such that they intersect for a given W_{n_i} and do not intersect for the others. Moreover, for any subset $\{n_i\}$ of $\{n_1, \ldots, n_k\}$, there exits two of lines, which intersect in $W_{\{n_i\}}$, but not in $\{n_1, \ldots, n_k\} \setminus \{n_i\}$.

Here are some examples. Let $\begin{cases} y = x +_n 2 \\ y = 4 \times_n x +_n 1 \end{cases}$ be the two lines, then solving this system, we get $1 = 3 \times_n x$ which has no solution for any n. Next, consider $\begin{cases} y = x +_n 1 \\ y = 3 \times_n x \end{cases}$. Solving the system, we get $x = 0.5$ and $y = 1.5$, so the two lines intersect for any W_n. Now consider, $\begin{cases} y = 1.7 \times_2 x +_2 0.8 \\ y = 0.1 \times_2 x +_2 1.8 \end{cases}$. To find this system's solution, consider its usual solution viewed in W_2, namely $(0.64, 1.86)$. Indeed, it is the solution of the given system. Now, when the solution is viewed in W_4, which is $(0.625, 1.8625)$, we also see that it is the solution, so the lines intersect for both $n = 2, 4$. For the next example, we consider the following two lines $\begin{cases} y = 31.85 \times_2 x +_2 1.28 \\ y = 7.41 \times_2 x +_2 7.12 \end{cases}$ and consider their usual solution $(0.24, 8.88)$. By simply plugging in these values into the system, we see that they satisfy it. However, when the usual solution is viewed in W_4, we get $(0.2389, 8.8878)$ and it does not satisfy the system, however, even if we vary the $x-$coordinate's last decimal by unit increments, we find that there can be no solution for this system.

We now move to the discussion of Lobachevsky's and Riemannian geometries. Fix the $x-$axis, l_0, and pick a point on the $y-$axis, say $(0, b)$. Then we have the following theorem: the line $y = kx + b$ is parallel to l_0 in Lobachevsky sense iff $|b| \geq 1$, and in case $|b| < 1$, we would only have parallel lines in Euclidean sense.

In the new geometry however, there are many lines (not just in Lobachevsky or Euclidean sense) which do not intersect l_0 but go through the point $(0,b)$. For example, consider $\begin{cases} y = 3 \times_n x +_n b \\ y = 0 \end{cases}$,

then $3 \times_n x = -b$ and then, for instance, for $b = 1$, the system will not have a solution for any n.

Obviously, for $b \neq 0$ we can always find (more than one) k, such that $\begin{cases} y = k \times_n x +_n b \\ y = 0 \end{cases}$ does not have no solution. Consider W_2, then, in particular, for $b = 1, 1.01, \ldots, 1.98$, we find that $k = 0.01$.

In fact, we have the following table for parallel lines:

b	k	Notes
0, 0.01,…, 0.99	0	Then there exits a unique line going through $(0,b)$ parallel to l_0, which is, in fact, parallel in Euclidean sense.
1, 1.01, …, 1.98	0.01	The lines going through $(0,b)$ are parallel to l_0 in Lobachevsky sense, and the lines for various values of b are parallel to each other in Euclidean sense.
1.99, 1.00, …, 2.97	0.02	...
2.98, 2.99,…, 3.96	0.03	...
…	…	...

This table was compiled using the fact that not intersecting the $x-$axis implies that the line gets as close as possible to l_0 in W_2, i.e. contains the point $(-99.99, 0.01)$ or $(-99.99, 0.02)$, etc.

Next, we consider Riemannian geometry, not as it is usually constructed, but as follows. Let us consider everything now with respect to W_2 and consider a unit cube centered at zero. Then the Riemannian line is an intersection of a plane containing the origin with faces of the cube. In the usual Riemannian geometry, any two such lines would intersect in two points, in the new geometry, however, the intersections may contain no, two, four, and twenty points. See Appendices 19-22. Also, for further numerical examples see Appendix 23.

The classical Riemannian geometry (a line is an intersection of a plane containing the origin with the unit sphere) also changes in the new light. Consider two planes

$\begin{cases} 0.2\times_2 x -_2 0.1\times_2 y -_2 10\times_2 z = 0 \\ 0.3\times_2 x -_2 0.2\times_2 y -_2 36\times_2 z = 0 \end{cases}$ and the sphere $x^2 +_2 y^2 +_2 z^2 = 1$. These three sets intersect in 100 points. Here is an example of two planes intersecting the sphere in exactly two points

$\begin{cases} x = 0 \\ y = 0 \\ x^2 +_2 y^2 +_2 z^2 = 0 \end{cases}$, which are $(0,0,\pm 1)$. The next system, $\begin{cases} x +_2 y -_2 4\times_2 z = 0 \\ y -_2 4\times_2 z = 0 \\ x^2 +_2 y^2 +_2 z^2 = 1 \end{cases}$ implies that

$\begin{cases} x = 0 \\ y = 4\times_2 z \\ 17\times_2 z^2 = 1 \end{cases}$ and hence no solution, since 17^{-1} does not exist (see Chapter 2). Hence there are two lines that do not intersect in the new Riemannian sense.

With this new geometry, the fractal geometry becomes meaningless; also the shoreline length paradox disappears. Explanations follow.