1. ARITHMETIC OPERATIONS IN OBSERVER'S MATHEMATICS

We consider a finite well-ordered system of observers, where each observer sees the real numbers as the set of all infinite decimal fractions. The observers are ordered by their level of "depth", i.e. each observer has a depth number (hence, we have the regular integer ordering), such that an observer with depth k sees that an observer with depth $n < k$ sees and deals (to be defined below) not with an infinite set of infinite decimal fractions, but, actually, with a finite set of finite decimal fractions. We call this set W_n, i.e. it is the set of all decimal fractions, such that there are at most n digits in the integer part and n digits in the decimal part of the fraction. Visually, an element in W_n looks like

$$\underbrace{\_\quad\cdots\quad\_}_{n}\,.\,\underbrace{\_\quad\cdots\quad\_}_{n}\,.$$

Moreover, an observer with a given depth is unaware (or can only assume the existence) of observers with larger depth values and for his purposes, he deals with "infinity". These observers are called *naive*, with the observer with the lowest depth number – the most naive. However, if there is an observer with a higher depth number, he sees that a given observer actually deals with a finite set of finite decimal fractions, and so on. Therefore, if we fix an observer, then this observer sees the sets W_{n_1} ,..., W_{n_k} with $n_1 < ... < n_k$ indicating the depth level, and realizes that the corresponding observers see and deal with infinity. When we talk about observers, we shall always have some fixed observer (called 'us') who oversees all others and realizes that they are naive. The "W_n-observer" is the abbreviation for somebody who *deals* with W_n while thinking that he deals with infinity.

We begin by defining sets W_n which consist of all finite decimal fractions such that there are at most n digits in the integer part and at most n digits in the decimal part. That is, the set W_n contains all elements of the form $a = a_0.a_1...a_n$ where the integer part can be written as $a_0 = b_{n-1}...b_0$, where $b_{n-1}, ..., b_0, a_1, ..., a_n \in \{0, 1, ..., 9\}$. Now, given $c = c_0.c_1...c_n$, $d = d_0.d_1...d_n \in W_n$ we endow W_n with the following arithmetic $(+_n, -_n, \times_n, \div_n)$ - from W_m - observer point of view $(m > n)$:

Definition 1.1. *Addition and subtraction*

$$c \pm_n d = \begin{cases} c \pm d, \ \textit{if}\ c \pm d \in W_n \\ \textit{not defined, if}\ c \pm d \notin W_n \end{cases}$$

where $c \pm d$ is the standard addition and subtraction, and we write $((\ldots(f_1 +_n f_2)\ldots) +_n f_N) = \sum_{i=1}^{N}{}^n f_i$ for $f_1, ..., f_N$ iff the contents of any parenthesis are in W_n, $f_1, \ldots, f_N \in W_n$.

Definition 1.2. *Multiplication*

$$c \times_n d = \sum_{k=0}^{n}{}^n \sum_{m=0}^{n-k}{}^n 0.\underbrace{0...0}_{k-1}c_k \cdot 0.\underbrace{0...0}_{m-1}d_m$$

where $c, d \geq 0$, $c_0 \cdot d_0 \in W_n$, $0.\underbrace{0...0}_{k-1}c_k \cdot 0.\underbrace{0...0}_{m-1}d_m$ is the standard product, and $k = m = 0$ means that $0.\underbrace{0...0}_{k-1}c_k = c_0$ and $0.\underbrace{0...0}_{m-1}d_m = d_0$. If either $c < 0$ or $d < 0$, then we compute $|c| \times_n |d|$ and *define $c \times_n d = \pm |c| \times_n |d|$, where the sign $\pm$ is defined as usual. Note, if the content of at least one parentheses (in previous formula) is not in W_n, then $c \times_n d$ is not defined.*

DEFINITION 1.3. *Division*

$$c \div_n d = \begin{cases} r, \text{ if } \exists\, r \in W_n \quad r \times_n d = c \\ \text{not defined, if no such } r \text{ exists} \end{cases}$$

Let $n = 2$, so we are in W_2. Here are some examples of elements of W_2: $3.14, -99, 0.1 \in W_2$ and $0.115, 123.9, -100000 \notin W_2$. Now, the examples of arithmetic:

$$2.08 +_2 11.9 = 13.98$$

$$(-2.08) +_2 11.9 = 9.82$$

$$80 +_2 24 = \text{not defined}$$

$$21.36 -_2 0.87 = 20.49$$

$$1.36 -_2 16.95 = -15.59$$

$$1.36 -_2 (-99.95) = \text{not defined}$$

$$11 \times_2 8 = 88$$

$$(-5) \times_2 19 = -95$$

$$11 \times_2 12 = \text{not defined}$$

$$3.41 \times_2 2.64 = 8.98$$

$$3.41 \times_2 (-2.64) = -8.98$$

$$3.41 \times_2 42.64 = \text{not defined}$$

$$99.41 \times_2 1.64 = \text{not defined}$$

$$0.85 \times_2 0.02 = 0$$

$$80 \div_2 4 = 20$$

$$1 \div_n 0.5 = \{2, 2.01, 2.02, 2.04, 2.05, 2.06, 2.07, 2.08, 2.09\}$$

- we get 10 different r's

$$1 \div_n 3 = \text{not defined}$$

(since no r exists). In case $p > q$, $\star \to \infty$ for W_q-observer means $\star \to 10^q$ for W_p-observer, and $\star \to 0$ for W_q-observer means $\star \to 10^{-q}$ for W_p-observer.

Here we provide some basic examples to illustrate what might happen.

1. Additive associativity fails:$(x +_n y) +_n z \neq x +_n (y +_n z)$, e.g. let 10, 95, $-35 \in W_2$, then $10 +_2 95 \notin W_2$, hence $(10 +_2 95) +_2 (-35) \notin W_2$, but $10 + ({}_2 95 +_2 (-35)) = 70 \in W_2$;

2. Multiplicative associativity fails:$(x \times_n y) \times_n z \neq x \times_n (y \times_n z)$, e.g. let 50.12, 0.85, and $0.61 \in W_2$, then $50.12 \times_2 0.85 = (50 + 0.1 + 0.02) \cdot (0.8 + 0.05) = 40 + 2.5 + 0.08 = 42.58$, and

$(50.12 \times_2 0.85) \times_2 0.61 = (42 + 0.5 + 0.08) \cdot (0.6 + 0.01) = 25.2 + 0.42 + 0.3 = 25.65$, whereas $0.85 \times_2 0.61 = (0.8 + 0.05) \cdot (0.6 + 0.01) = 0.48$ and $50.12 \times_2 (0.85 \times_2 0.61) = (50 + 0.1 + 0.02) \cdot (0.4 + 0.08) = 20 + 4 + 0.04 = 24.04$;

3. Distributivity fails: $x \times_n (y +_n z) \neq x \times_n y +_n x \times_n z$, e.g. let $1.81,\ 0.74,\ 0.53 \in W_2$, then $0.74 +_2 0.53 = 1.27$ and $1.81 \times_2 (0.74 +_2 0.53) = (1 + 0.8 + 0.01) \cdot (1 + 0.2 + 0.07) = 1 + 0.2 + 0.07 + 0.8 + 0.16 + 0.01 = 2.24$, whereas $1.81 \times_2 0.74 = (1 + 0.8 + 0.01) \cdot (0.7 + 0.04) = 0.7 + 0.04 + 0.56 = 1.3$ and $1.81 \times_2 0.53 = (1 + 0.8 + 0.01) \cdot (0.5 + 0.03) = 0.5 + 0.03 + 0.4 = 0.93$, so that $1.81 \times_2 0.74 +_2 1.81 \times_2 0.53 = 2.23$;

4. Lack of the distribution law leads to the following result:

The statement "$x|y$ and $x|z \quad \Rightarrow \quad x|\,(y+z)$" is false. Here $x|y \quad \Leftrightarrow \exists r : x \times_n r = y$. Assume that $x|y$ and $x|z$, what we want to show is equivalent to showing that $y + z \neq x \times_n (r_1 +_n r_2)$ for some x, y, z, r_1 and r_2. Let $x = 0.17$, $r_1 = 0.85$, $r_1 = 0.63$, $y = 0.17 \times_2 0.85 = 0.(0.1 + 0.07) \cdot (0.8 + 0.05) = 0.08$ and $z = 0.17 \times_2 0.63 = 0.(0.1 + 0.07) \cdot (0.6 + 0.03) = 0.06$. Then $y + z = 0.14$, but $r_1 + r_2 = 1.48$ and $0.17 \times_2 1.48 = (0.1 + 0.07) \cdot (1 + 0.4 + 0.08) = 0.1 + 0.04 + 0.07 = 0.21$. In fact, x is not divizor of $y + z$. This is because if we let $0.17 \times_2 0.9 = (0.1 + 0.07) \cdot 0.9 = 0.09 < 0.14$ and $0.17 \times_2 0.99 = (0.1 + 0.07) \cdot (0.9 + 0.09) = 0.09 < 0.14$, but $0.17 \times_2 1 = 0.17 > 0.14$.

5. Multiplicative inverses do not necessarily exist, or if they do, they are not necessarily unique in W_n. Here are some examples: let $2 \in W_n$, then $0.5 \in W_2$ is the unique inverse of 2 for any W_n. On the other hand, 3 will not have an inverse in any W_n. Now, let $2^{-1} = 0.5$, then $(0.5)^{-1}$is actually the following set $\{2, 2.01, 2.02, 2.03, 2.04, 2.05, 2.06, 2.07, 2.08, 2.09\} \in W_2$. Therefore, $(2^{-1})^{-1}$ is not necessarily 2, hence all we can claim is that if x^{-1} exists, then $x \in \left\{(x^{-1})^{-1}\right\}$. Further, if an inverse of an element exists in W_n, it does not necessarily exist in W_m for $m \neq n$, independent of the order of $m$and n, e.g. if $0.91 \in W_2$, then $(0.91)^{-1} = \{1.1, 1.11, 1.12, 1.13, 1.14, 1.15, 1.16, 1.17, 1.18, 1.19\} \in W_2$, but $(0.91)^{-1} \notin W_4$, on the other hand, $16^{-1} = 0.0625 \in W_4$, but $16^{-1} \notin W_2$.

6. Square roots do not necessarily exist. Some examples are, if $4 \in W_n$, then $\sqrt{4} = 2$ for any n and $\sqrt{3}$ does not exist in $n = 2$. To show that, consider $1.75 \times_2 1.75 = (1 + 0.7 + 0.05) \cdot (1 + 0.7 + 0.05) = 1 + 0.7 + 0.05 + 0.7 + 0.49 + 0.05 = 2.99$ and

$$1.76 \times_2 1.76 = (1 + 0.7 + 0.06) \cdot (1 + 0.7 + 0.06) = 1 + 0.7 + 0.06 + 0.7 + 0.49 + 0.06 = 3.01.$$

Further, if a square root of an element exists in W_n, it does not necessarily exist in W_m for $m \neq n$, independent of the order of m and n, e.g. $\sqrt{2} = 1.42 \in W_2$, since $1.42 \times_2 1.42 = (1 + 0.4 + 0.02) \cdot (1 + 0.4 + 0.02) = 1 + 0.4 + 0.02 + 0.4 + 0.16 + 0.02 = 2$, but $\sqrt{2} \notin V_4$,

since $1.4143 \times_4 1.4143 = (1 + 0.4 + 0.01 + 0.004 + 0.0003) \cdot (1 + 0.4 + 0.01 + 0.004 + 0.0003) = 1.9999$

and $1.4144 \times_4 1.4144 = 2.0001$. Also, $\sqrt{1.01} = 1.005 \in W_4$, since $1.005 \times_4 1.005 = (1 + 0.005) \cdot (1 + 0.005) = 1 + 0.005 + 0.005 = 1.01$, but $\sqrt{1.01} \notin W_2$, since $1 \times_2 1 = 1$ and $1.01 \times_2 1.01 = (1 + 0.01) \cdot (1 + 0.01) = 1 + 0.01 + 0.01 = 1.02$.

Next, some basic theorems can be stated for W_n:

1. Any W_n has zero divisors: $0.\underbrace{0...0}_{n-1}1 \times_n 0.\underbrace{0...0}_{n-1}1 = 0$;
2. If $p \in W_n$ with $p \neq 2, 5$ a prime in the usual sense, then $p^{-1} \notin W_n$ for any W_n;
3. $\forall x, y \in W_n$ with $x, y \geq 0$, $x - y \in W_n$.
4. If $x, y, t, u \in W_n$ and $x \geq t \geq 0$ and $y \geq u \geq 0$ and $x \times_n y \in W_n$, then $t \times_n u \in W_n$ and $x \times_n y \geq t \times_n u$
5. If given $a \in W_n$ such that there is a unique $a^{-1} \in W_n$, then $|a| \geq 1$;
6. If $|a| < 1$ and a^{-1} exists, then $|\{a^{-1}\}| > 1$;
7. If $|\{a^{-1}\}| > 1$, then $|a| < 1$.

Let's consider now additional valuable properties of introduced arithmetic.

1. Standard multiplications identities become wrong, for example

$$(x+y)^2 \neq x^2 + 2\,(xy) + y^2$$

.

We have

THEOREM 1.4. *$P\left((a +_n b) \times_n (a +_n b) = (a \times_n a +_n 2 \times_n (a \times_n b)) +_n b \times_n b\right) < 1$, where P is a probability.* We can see a proof below. Let $n = 2$. Then

1. Left side of equality is $(1.32 +_2 2.43) \times_2 (1.32 +_2 2.43) = 3.75 \times_2 3.75 = 13.99$, right side consists from two parts. First, $1.32 \times_2 1.32 = 1.73$; second, $2 \times_2 (1.32 \times_2 2.43) = 6.38$, and finally $2.43 \times_2 2.43 = 5.88$. That means $1.73 +_2 6.38) +_2 5.88 = 13.99$. I.e left side equals to right. But now let's consider the following calculations.
2. Left side of equality is $(1.32 +_2 2.79) \times_2 (1.32 +_2 2.79) = 4.11 \times_2 4.12 = 16.89$, right side consists from two parts. First, $1.32 \times_2 1.32 = 1.73$; second, $2 \times_2 (1.32 \times_2 2.79) = 7.28$, and finally $2.79 \times_2 2.79 = 7.65$. That means $1.73 +_2 7.28 +_2 7.65 = 16.66$. I.e left side does not equal to right.

Let's consider now a random variable

$$\delta_1 = (a +_n b) \times_n (a +_n b) - ((a \times_n a +_n 2 \times_n (a \times_n b)) +_n (b \times_n b))$$

where $a, b \geq 0$, and δ_1, and all elements of right side belong to W_n. Let's $n = 2$. Using direct calculation we can build $F_1(x)$ - distribution function of δ_1, where

$$F_1(x) = P(\delta_1 < x)$$

, P is a probability. Graph of $F_1(x)$ you can see on Fig. 1.

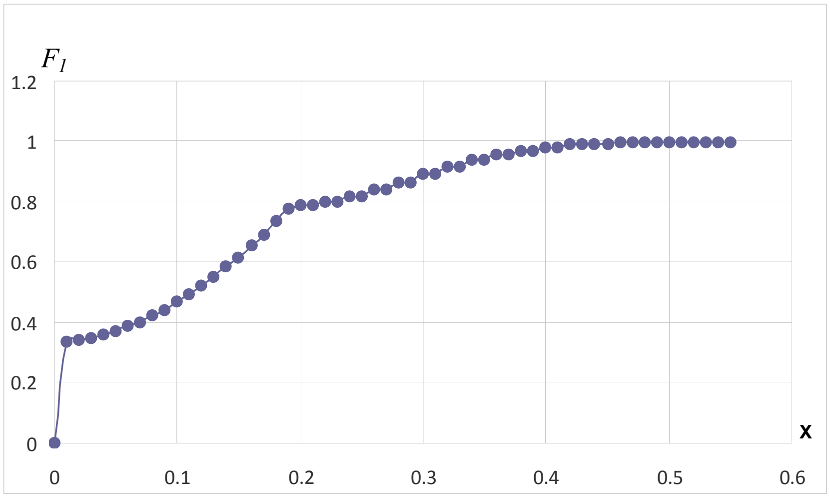

Figure 1. Graph F_1.

General proof for W_n you can see below. If a, b non-negative integers in W_n and $(a +_n b) \times_n (a +_n b) \in W_n$, then $\delta_1 = 0$. let's consider now $a = 0.\underbrace{9\ldots9}_n$ and $b = 0.\underbrace{0\ldots08}_n$. Then $a +_n b = 1.\underbrace{0\ldots07}_n$ and $(a +_n b) \times_n (a +_n b) = 1.\underbrace{0\ldots07}_n \times_n 1.\underbrace{0\ldots07}_n = 1.\underbrace{0\ldots014}_n$, but $a \times_n a < 1$, $b \times_n b = 0$, and $2 \times_n (a \times_n b) = 0$. I.e. $\delta_1 \neq 0$.

2. We have also the following theorem.

THEOREM 1.5.

$$P(c \times_n (a +_n b) = c \times_n a +_n c \times_n b) < 1$$

, where P is a probability. Below you can see a proof. Let's $n = 2$. Then

1. Left side of equality is $2 \times_2 (3 +_2 6) = 2 \times_2 9 = 18$, right side consists from two parts. First, $2 \times_2 3 = 6$, then $2 \times_2 6 = 12$ and $6 +_2 12 = 18$ I.e. left side equals to right. But go to next calculations.

2. Left side of equality is $2.41 \times_2 (3.14 +_2 0.58) = 2.41 \times_2 3.72 = 8.95$, right side consists from two parts. First, $2.41 \times_2 3.14 = 7.55$, then $2.41 \times_2 0.58 = 1.36$ и $7.55 +_2 1.36 = 8.91$. I.e. left side does not equal to right.

Let's consider a random variable

$$\delta_2 = c \times_n (a +_n b) -_n (c \times_n a +_n c \times_n b)$$

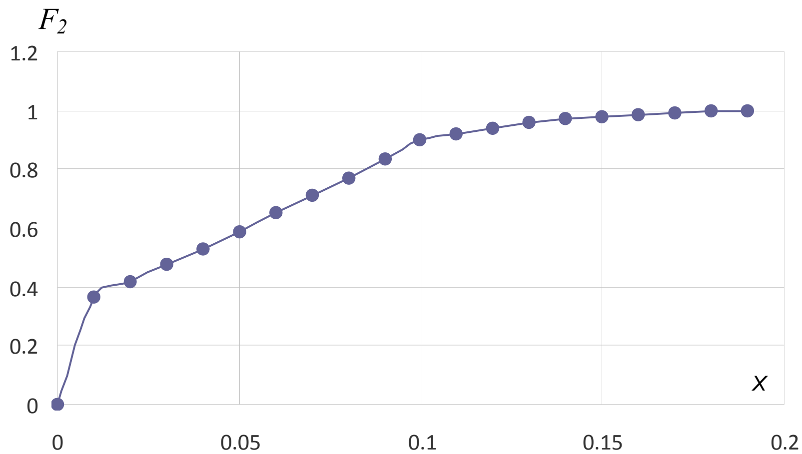

Figure 2. Graph F_2.

where $a, b, c \geq 0$, and δ_2 and all elements of right side belong to W_n. Let's $n = 2$. Using direct calculations we can build $F_2(x)$ - distribution function of δ_2, where $F_2(x) = P(\delta_2 < x)$, where P is a probability. Graph of $F_2(x)$ you can see on Fig. 2.

General proof for W_n you can see below. If a, b, c - non-negative integers in W_n and $a \times_n (b \times_n c) \in W_n$, then $\delta_2 = 0$. Let's consider now $a = 2$, $b = 0.\underbrace{9\ldots9}_{n}$ and $c = 0.\underbrace{0\ldots01}_{n}$. Then $b \times_n c = 0$, $a \times_n (b \times_n c) = 0$, $a \times_n b = 1.\underbrace{9\ldots98}_{n}$, and $(a \times_n b) \times_n c = 0.\underbrace{0\ldots01}_{n}$. I.e. $\delta_2 \neq 0$.

3.

THEOREM 1.6. *Let's*

$$\delta_3 = c \times_n (a \times_n b) -_n (c \times_n a) \times_n b$$

. *Then* $0 < P(\delta_3 = 0) < 1$, *where* P *is a probability.*

You can see a proof of this theorem below. Let's $n = 2$. Then

1. Left side of this equality is $2 \times_2 (3 \times_2 6) = 2 \times_2 18 = 36$, right side consists from two parts . First, $2 \times_2 3 = 6$, then $6 \times_2 6 = 36$. I.e left side equals to right. But let's consider the following calculations.

2. Left side is $2.41 \times_2 (3.14 \times_2 0.58) = 2.41 \times_2 1.79 = 4.27$, for right side we get first , $2.41 \times_2 3.14 = 7.55$, then $7.55 \times_2 0.58 = 4.31$. And left side does not equal to right Let's consider a random variable

$$\delta_3 = c \times_n (a \times_n b) -_n (c \times_n a) \times_n b$$

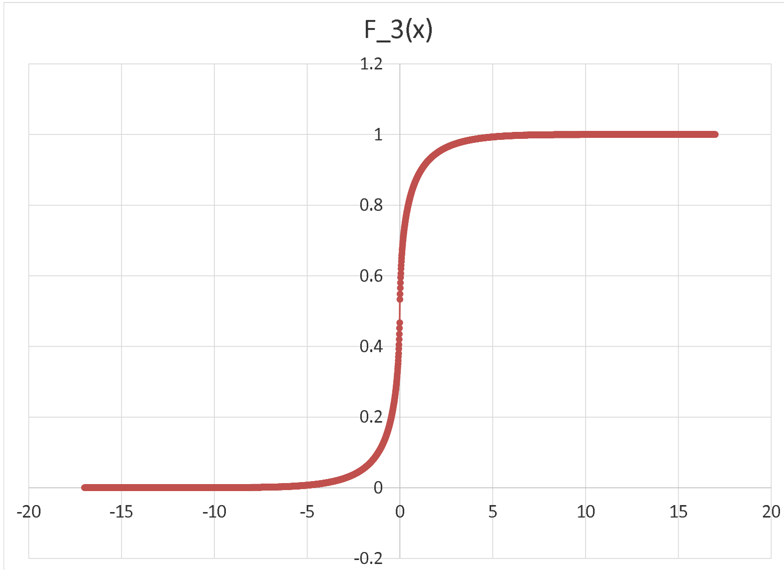

Figure 3. Graph F_3.

, where $a, b, c \geq 0$, and δ_3 and all elements of right side belong to W_n. If we take $n = 2$, then using direct calculations we can build $F_3(x)$ - distribution function of δ_3, where $F_3(x) = P(\delta_3 < x)$, and P is a probability. Graph of $F_3(x)$ you can see on Fig. 3. General proof for W_n you can see below. If a, b, c are non-negative integers in W_n and $c \times_n (a \times_n b) \in W_n$, then $\delta_3 = 0$. Let's consider $c = 2$, $a = 0.\underbrace{9\ldots99}_{n}$ и $b = 0.\underbrace{0\ldots01}_{n}$. Then

$$\delta_3 = 2 \times_n (0.9\ldots99 \times_n 0.0\ldots01) -_n (2 \times_n 0.9\ldots99) \times_n 0.0\ldots01 =$$
$$= 0 -_n 0.0\ldots01 = -0.0\ldots01 \neq 0$$