2. OBSERVER'S MATHEMATICS APPLICATIONS TO CLASSICAL MATHEMATICS PROBLEMS

2.1 Analogy of Fermat's, Mersenne's and Waring's Problems

We proved the following four Theorems:

THEOREM 2.1. *(Analogy of Fermat's Last Problem). For any integer n, $n \geq 2$, and for any integer m, $m \geq 3$, $m \in W_n$ (see below for the definition of W_n) there exist positive a, b, $c \in W_n$, such that $a^m +_n b^m = c^m$ (operation $+_n$ is defined below).*

For example, if $n = 2$ we can calculate that

$1^3 +_2 1^3 = 1.28^3$

$1^3 +_2 1.21^3 = 1.41^3$

$1.2^3 +_2 1.03^3 = 1.41^3$

Note that the main reason of cardinal difference between standard Mathematics and Observer's Mathematics results is the following. The negative solution of classical Fermat's problem requires Axiom of Choice to be valid. But in Observer's Mathematics this Axiom is invalid.

THEOREM 2.2. *(Analogy of Mersenne's numbers problem). There exist integers n, $k \geq 2$, Mersenne's numbers M_k, with $\{k, M_k\} \in W_n$, and positive $a \in W_n$, such that $M_k = a^2$.*

THEOREM 2.3. *(Analogy of Fermat's numbers problem). There exist integers n, $k \geq 2$, Fermat's numbers F_k, $\{k, F_k\} \in W_n$, and positive $a \in W_n$, such that $F_k = a^2$.*

THEOREM 2.4. *(Analogy of Waring's problem). For any integer k, $k \geq 2$, there exist integer n, $n \geq 2$, ($k \in W_n$) and some $x \in W_n$ such that any equality of the form $x = a_1^k + a_2^k + \ldots + a_l^k$ is not possible for any integer $l \in W_n$ and any positive numbers a_1, a_2, $\ldots$, $a_l \in W_n$.*

2.2 Analogy of Hilbert's Tenth Problem

First we are looking for the solution of equation

$$x^3 + y^3 + z^3 = 33$$

in W_n, $n \geq 2$, i.e. we have to find $x, y, z \in W_n$ such that $((x \times_n x) \times_n x)$, $((y \times_n y) \times_n y)$, $((z \times_n z) \times_n z)$, $((x \times_n x) \times_n x) +_n ((y \times_n y) \times_n y) \in Wn$, and $(((x \times_n x) \times_n x) +_n ((y \times_n y) \times_n y)) +_n ((z \times_n z) \times_n z) = 33$. We provide several solutions below:

1. For $n = 2$, the solutions are:
   (a) $\{1.72, 1, 3\}$,
   (b) $\{-1.28, 2, 3\}$,
   (c) $\{2.37, 1.55, 2.54\}$.
2. For $n = 3$, the solutions are:

(a) $\{3.208, 0, 0\}$,
(b) $\{3.208, y, -y\}$ for any $y \in W_3$ such that $(y \times_3 y) \times_3 y$, $(y \times_3 y) \times_3 y + 33 \in W_3$,
(c) $\{2.887, 1, 2\}$.

3. For $n = 4$, a possible solution is: $\{2.4102, -2, 3\}$.

4. For $n = 5$, a possible solution is: $\{4.12129, 3, -4\}$.

5. For $n = 6$, the solutions are:

   (a) $\{2.8845, 1, 2\}$,
   (b) $\{1.709981, 1, 3\}$.

6. For $n = 9$, a possible solution is: $\{2.571281595, -1, -1\}$.

7. For $n = 10$, a possible solution is: $\{3.8929964162, 1, -3\}$.

8. For $n = 11$, a possible solution is: $\{3.89299641591, 1, -3\}$.

9. For $n = 12$, a possible solution is: $\{3.659305710025, -2, -2\}$.

10. For $n = 13$, the solutions are:

    (a) $\{2.9240177382132, 0, 2\}$,
    (b) $\{2.9240177382132, 2, 0\}$.

11. For $n = 14$, the solutions are:

    (a) $\{4.08165510191737, -2, -3\}$,
    (b) $\{4.71769398031657, -2, -4\}$.

12. For $n = 15$, the solutions are:

    (a) $\{2.410142264175234, -2, 3\}$,
    (b) $\{1.259921049894891, 2, 3\}$,
    (c) $\{4.081655101917351, -2, -3\}$.

In W_n, from the W_n-observer's point of view (W_n-observer is "naive" in W_n), Hilbert's Tenth Problem is formulated classically: "Is there an algorithm that takes as input a multivariable polynomial $f(x_1, \ldots, x_k)$ with integer coefficients and outputs YES or NO according to whether there exist integers $a_1, \ldots, a_k$ such that $f(a_1, \ldots, a_k) = 0$." And W_n-observer as "naive" one has and understands proof, which Yu. Matiyasevich based on works of M. Davis, H. Putham, and J. Robinson made in 1970, and shown that no such algorithm exists. Consider now what does it mean from W_m-observer's point of view ($m > n$).

First we address the question "what is a polynomial in W_n" from the point of view of W_m-observer, with $m > n$?

DEFINITION 2.5. *Multivariable (k - variables) polynomial $f(x_1,\ldots,x_k)$ with degree q in W_n is given by:* $\sum_{p=0}^{q}{}^{n}\sum_{i_1+\ldots+i_k=p}{}^{n}a_{i_1\ldots i_k}\times_n(\ldots(\ldots\underbrace{(x_1\times_n x_1)\times_n\ldots)\times_n x_1)}_{i_1}\times_n\ldots$

$\times_n(\ldots\underbrace{(x_k\times_n x_k)\times_n\ldots)\times_n x_k)}_{i_k}$ *where $k\in N\cap W_n$, $q,i_1,\ldots,i_k\in(N\cup 0)\cap W_n$, with N - the set of all positive integers, and $a_{i_1\ldots i_k},x_1,\ldots,x_k$ and entries of all parentheses are in W_n.*

THEOREM 2.6. *For any positive integers $m,n,k\in W_n$, $n\in W_m$, $m>log_{10}(1+(2\cdot 10^{2n}-1)^k)$, from the point of view of the W_m-observer, there is an algorithm that takes as input a multivariable polynomial $f(x_1,\ldots,x_k)$ of degree q in W_n and outputs YES or NO according to whether there exist $a_1,\ldots,a_k\in W_n$ such that $f(a_1,\ldots,a_k)=0$.*

Note, that, for example, for $n=2$ and $k=3$, this problem has negative solution from the point of view of not only W_2-observer, but also for W_3,W_4,...,W_{12}-observers and only from the point of view of W_m-observer with $m\geq 13$ this problem has positive solution.

Therefore, Hilbert's tenth problem in Observer's Mathematics has positive solution. We think that Hilbert expected a positive answer for his tenth problem. Note, that the main reason of cardinal difference between standard Mathematics and Observer's Mathematics results is the following. The negative solution of classical tenth problem requires Axiom of Choice to be valid. But in Observer's Mathematics this Axiom is invalid.

2.3 Lehmer's Number in Observer's Mathematics

Lehmer's number, $\alpha\approx= 1.17628$, is the largest real root of the polynomial

$$f(x)=x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1$$

. This number appears in various contexts in number theory and topology as the (sometimes conjectural) answer to natural questions involving notions of "minimality" and "small complexity".

A Salem polynomial is a monic irreducible reciprocal polynomial $\phi(x)$ in $\mathbb{Z}[x]$ such that $\phi(x)=0$ has exactly two real roots $\alpha>1$ and $1/\alpha$ off the unit circle $S^1:=\{z\in\mathbb{C}||z|\}$. Is is then of even degree. A Salem number is the unique real root $\alpha>1$. In other words, a Salem number of degree $2n$ is a real algebraic integer $\alpha>1$ whose Galois conjugates consist of $1/\alpha$ and $2n-2$ imaginary numbers on S^1.

There are 47 known Salem numbers less than 1.3. Of these, 45 exhaust the possibilities with $\alpha<1.3$ and degree $d<42$. Of these, merely 6 have degree $d<12$. Of these 6, we noted that all but one solve equation of the very simple form $x^{4+m}=\frac{Q(1/x)}{Q(x)}$ with $m>0$ and $Q(x)=x^3-x-1$. The case $m=1$ gives Lehmer's number field. The minimal polynomials of the first five Salem numbers in this family are

$$P_1(\alpha)=\alpha^{10}+\alpha^9-\alpha^7-\alpha^6-\alpha^5-\alpha^4-\alpha^3+\alpha+1$$

$$P_2(\alpha)=\alpha^{10}-\alpha^7-\alpha^5-\alpha^3+1$$

$$P_3(\alpha) = \alpha^{10} - \alpha^8 - \alpha^5 - \alpha^2 + 1$$
$$P_4(\alpha) = \alpha^8 - \alpha^5 - \alpha^4 - \alpha^3 + 1$$
$$P_5(\alpha) = \alpha^{10} - \alpha^8 - \alpha^7 + \alpha^5 - \alpha^3 - \alpha^2 + 1$$

with approximate numerical roots - and hence Mahler measures - given by

$$\alpha_1 = 1.17628081825991750654407033784\ldots$$

$$\alpha_2 = 1.23039143440722470279017793 89\ldots$$
$$\alpha_3 = 1.26123096113713885194667150 30\ldots$$
$$\alpha_4 = 1.28063815626775759670190253 27\ldots$$
$$\alpha_5 = 1.29348595312545410651990988 37\ldots$$

Prior to understanding the Lehmer-Salem numbers in the setting of Observer's Mathematics, we first address the question "what is a polynomial in W_n?".

DEFINITION 2.7. *Polynomial $f(x)$ with degree q in W_n is given by the following formula:*

$$f(x) = \sum_{p=0}^{q}{}^{n} a_p \times_n (\ldots(x \times_n x) \times_n x) \times_n \ldots \times_n x)$$

where q, a_p, x and entries of all parentheses are in W_n, $p = 0, 1, \ldots, q$.

Note that exponent is not an associative operation. For example, for $n = 2$, we have $1.49 \times_2 1.49 = 2.14$, $1.49 \times_2 2.14 = 3.16$, $1.49 \times_2 3.16 = 4.67$, i.e., $((1.49 \times_2 1.49) \times_2 1.49) \times_2 1.49 = 4.67$ while $(1.49 \times_2 1.49) \times_2 (1.49 \times_2 1.49) = 4.57$.

In W_n we can define a root of a polynomial $f(x) \in W_n$ as the number $x_0 \in W_n$ such that $|f(x)| \leq 0.\underbrace{0\ldots01}_{n}$.

THEOREM 2.8. *There are some $n \in \mathbb{N}$ such that the minimal polynomial of the first five Salem numbers have the roots in W_n. Note that we consider $P_1(\alpha), P_3(\alpha), P_4(\alpha)$ as the polynomial in W_n in the sense of the definition above, i.e., we understand that $x^k = (\ldots\underbrace{(x \times_n x) \times_n \ldots \times_n x)}_{k}$ and $\sum = \sum^n$.*

2.4 Euler Brick and Perfect Cuboid problems

An Euler Brick is just a cuboid, or a rectangular box, in which all of the edges (length, depth, and height) have integer dimensions and in which the diagonals on all three sides are also integers. So if the length, depth and height are a, b, and c respectively, then a, b, and c are integers, as are the quantities $\sqrt{a^2+b^2}$, $\sqrt{b^2+c^2}$ and $\sqrt{c^2+a^2}$. The unsolved problem is to find a four dimensional Euler Brick, in which the four sides a, b, c, and d are integers, as are the six face diagonals $\sqrt{a^2+b^2}$, $\sqrt{a^2+c^2}$, $\sqrt{a^2+d^2}$, $\sqrt{b^2+c^2}$, $\sqrt{b^2+d^2}$ and $\sqrt{c^2+d^2}$, or prove that such a cuboid cannot exist.

We reformulate $4D$ Euler Brick problem for Observer's Mathematics in the following way. To find some positive integer n and a $4D$ cuboid, in which the four sides a, b, c, d are integers in W_n, and the six face diagonals $\sqrt{a^2+b^2}$, $\sqrt{a^2+c^2}$, $\sqrt{a^2+d^2}$, $\sqrt{b^2+c^2}$, $\sqrt{b^2+d^2}$ and $\sqrt{c^2+d^2}$ are also in W_n, or prove that such cuboid cannot exist.

THEOREM 2.9. *If $a=b=c=d=1$, then the following condition holds true in W_2*

$$\sqrt{a^2+b^2}=\sqrt{a^2+c^2}=\sqrt{a^2+d^2}=\sqrt{b^2+c^2}=\sqrt{b^2+d^2}=\sqrt{c^2+d^2}=1.42\in W_2$$

Also, the following condition holds true in W_3:

$$\sqrt{a^2+b^2}=\sqrt{a^2+c^2}=\sqrt{a^2+d^2}=\sqrt{b^2+c^2}=\sqrt{b^2+d^2}=\sqrt{c^2+d^2}=1.416\in W_3$$

The above theorem implies that this problem has a positive solution in Observer's Mathematics.

Another unsolved problem is to find a perfect cuboid, which is an Euler Brick in which the space diagonal, that is, the distance from any corner to its opposite corner, given by the formula $\sqrt{a^2+b^2+c^2}$, is also an integer, or prove that such a cuboid cannot exist.

We reformulate perfect cuboid problem for Observer's Mathematics in the following way. To find some positive integer n and a perfect cuboid, in which the three sides a, b, c are integers in W_n, and the three face diagonals $\sqrt{a^2+b^2}$, $\sqrt{a^2+c^2}$ and $\sqrt{b^2+c^2}\in W_n$, and in which the space diagonal, that is, the distance from any corner to its opposite corner, given by the formula $\sqrt{a^2+b^2+c^2}\in W_n$, or prove that such a cuboid cannot exist.

THEOREM 2.10. *If $a=b=c=1$, then the following condition holds true in W_2:*

$$\sqrt{a^2+b^2}=\sqrt{a^2+c^2}=\sqrt{b^2+c^2}=\sqrt{2}=1.42\in W_2$$

However, $\sqrt{a^2+b^2+c^2}=\sqrt{3}$ does not exist. Also, the following condition holds true in W_3

$$\sqrt{a^2+b^2}=\sqrt{a^2+c^2}=\sqrt{b^2+c^2}=\sqrt{2}=1.416\in W_3$$

And $\sqrt{a^2+b^2+c^2}=\sqrt{3}=1.734\in W_3$ The above theorem implies that this problem has a positive solution in Observer's Mathematics.

2.5 Square Peg Problem

Every continuous simple closed curve in the plane defined by

$$\gamma: S^1 \to R^2$$

contains four points that are the vertices of a square. Is it true or not true? Let's take the Observer's Math point of view. Let's consider the identity $\gamma: S^1 \to S^2$ givne by $\gamma(x,y)=(x,y)$ with $(x,y)\in S^1$. In this case one of vertices (if such square exists) has to be the intersection of line $y=x$ and circle $x^2+_n y^2=R^2$. Let's note we assume that circle has a center in $(0,0)$ and square has edges parallel to coordinate axes. In classical math for any square with vertices (x_0,x_0), $(-x_0,x_0)$, $(x_0,-x_0)$, and $(-x_0,-x_0)$ the circle containing these points always exists.

Theorem 2.11. . *In Observer's Mathematics, for any square with vertices* (x_0, x_0), $(-x_0, x_0)$, $(x_0, -x_0)$, *and* $(-x_0, -x_0)$, *the circle containing these points does not always exist.*

Proof. If x_0, $R \in Wn$, $x_0 = 1, 2 = R^2$, then we have

$$n = 2 \to R = 1.42$$

$$n = 3 \to R = 1.416$$
$$n = 4 \to R \text{ does not exist}$$
$$n = 5 \to R = 1.41423$$
$$n = 6 \to R = 1.414216$$
$$n = 7 \to R = 1.4142139$$
$$n = 8 \to R \text{ does not exist}$$
$$n = 9 \to R = 1.414213567$$
$$n = 10 \to R \text{ does not exist}$$

If x_0, $R \in W_n$, $x_0 = 2$, $8 = R^2$, than we have

$$n = 2 \to R = 2.84$$

$$n = 3 \to R \text{ does not exist}$$
$$n = 4 \to R = 2.8287$$
$$n = 5 \to R = 2.82846$$
$$n = 6 \to R \text{ does not exist}$$
$$n = 7 \to R = 2.8284274$$
$$n = 8 \to R = 2.82842717$$
$$n = 9 \to R = 2.828427129$$
$$n = 10 \to R \text{ does not exist}$$

If x_0, $R \in W_n$, $x_0 = 3$, $18 = R^2$, than we have

$$n = 2 \to R \text{ does not exist}$$

$$n = 3 \to R = 4.243$$
$$n = 4 \to R = 4.2427$$
$$n = 5 \to R = 4.24265$$
$$n = 6 \to R \text{ does not exist}$$
$$n = 7 \to R = 4.2426408$$
$$n = 8 \to R = 4.2426407$$

$$n = 9 \to R = 4.242640689$$
$$n = 10 \to R \text{ does not exist}$$

If x_0 , $R \in W_n$, $x_0 = 4$, $32 = R^2$, than we have

$$n = 2 \to R \text{ does not exist}$$
$$n = 3 \to R = 5.658$$
$$n = 4 \to R \text{ does not exist}$$
$$n = 5 \to R \text{ does not exist}$$
$$n = 6 \to R \text{ does not exist}$$
$$n = 7 \to R \text{ does not exist}$$
$$n = 8 \to R \text{ does not exist}$$
$$n = 9 \to R \text{ does not exist}$$
$$n = 10 \to R \text{ does not exist}$$

If x_0 , $R \in W_n$, $x_0 = 5$, $50 = R^2$, than we have

$$n = 2 \to R \text{ does not exist}$$
$$n = 3 \to R \text{ does not exist}$$
$$n = 4 \to R \text{ does not exist}$$
$$n = 5 \to R \text{ does not exist}$$
$$n = 6 \to R \text{ does not exist}$$
$$n = 7 \to R = 7.0710679$$
$$n = 8 \to R \text{ does not exist}$$
$$n = 9 \to R \text{ does not exist}$$
$$n = 10 \to R \text{ does not exist}$$

□

THEOREM 2.12. *In Observer's Mathematics for any circle the square with vertices* (x_0, x_0), $(-x_0, x_0)$, $(x_0, -x_0)$, *and* $(-x_0, -x_0)$ *does not always exist.*

Proof. If x_0, $R \in W_n$, $R = 1$, $2 \times_n x_0^2 = 1$, than we have

$$n = 2 \to x_0 \text{ does not exist}$$
$$n = 3 \to x_0 \text{ does not exist}$$
$$n = 4 \to x_0 \text{ does not exist}$$
$$n = 5 \to x_0 \text{ does not exist}$$

$$n = 6 \rightarrow x_0 \text{ does not exist}$$
$$n = 7 \rightarrow x_0 \text{ does not exist}$$
$$n = 8 \rightarrow x_0 \text{ does not exist}$$
$$n = 9 \rightarrow x_0 = 0.70710679*$$

where $*$ means any digit $\in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

$$n = 10 \rightarrow x_0 \text{ does not exist}$$

□

That means that the Square Peg Problem has negative solution in Observer's Mathematics.

2.6 Classical geometric problem of angle trisection

Consider the equation

$$x^3 - 3x - 2\cos\alpha = 0$$

Note that if we take unit circle on the real plane $x^2 + y^2 = 1$ and put $\cos\alpha = z(z \in (0, 1))$ for some W_n, then $\sin\alpha$ may not exist. For example, in W_2, if

$$\cos\alpha = 0.6, 0.61, 0.62, 0.63, 0.64, 0.65, 0.66, 0.67, 0.68, 0.69$$

, then

$$\sin\alpha = 0.8, 0.81, 0.82, 0.83, 0.84, 0.85, 0.86, 0.87, 0.88, 0.89$$

though these 10 different $\sin\alpha$ values correspond to each cos a value. Also, If

$$\cos\alpha = 0.8, 0.81, 0.82, 0.83, 0.84, 0.85, 0.86, 0.87, 0.88, 0.89$$

then

$$\sin\alpha = 0.6, 0.61, 0.62, 0.63, 0.64, 0.65, 0.66, 0.67, 0.68, 0.69$$

, though these 10 different $\sin\alpha$ values correspond to each $\cos\alpha$ value. For any other possible positive values of $\cos\alpha$ in W_2 the $\sin\alpha$ does not exist.

THEOREM 2.13. *For any possible positive value of* $\cos\alpha$ *in* W_2 *equation* $x^3 - 3x - 2\cos\alpha = 0$ *does not have a solution in* W_2.

THEOREM 2.14. *For* $\cos\alpha = 0.492 \in W_3$, *in this case*

$$\sin\alpha = 0.88, 0.881, 0.882, 0.883, 0.884, 0.885, 0.886, 0.887, 0.888, 0.889$$

then the solution of equation $x^3 - 3x - 2\cos\alpha = 0$ *exists and it is* $x = 1.88$ *in* W_3.