4. TENSORS IN OBSERVER'S MATHEMATICS

Let's remind - we consider arithmetic operation in W_p from W_q- observer point of view ($p < q$).

4.1 Space $E_m W_n$

Let's consider Cartesian product of m copies of W_n: $E_m W_n = \underbrace{W_n \times \ldots \times W_n}_{m}$. We call "vector" any element from $E_m W_n$: $\mathbf{a} = (a_1, \ldots, a_m)$, $a_1, \ldots, a_m \in W_n$. If $\mathbf{a}$, $\mathbf{b} \in E_m W_n$, $\mathbf{a} = (a_1, \ldots, a_m)$, $\mathbf{b} = (b_1, \ldots, b_m)$, $\alpha \in W_n$, we define

$$\mathbf{a} +_n \mathbf{b} = (a_1 +_n b_1, \ldots, a_m +_n b_m)$$

if $a_1 +_n b_1, \ldots, a_m +_n b_m \in W_n$.

$$\alpha \times_n \mathbf{a} = (\alpha \times_n a_1, \ldots, \alpha \times_n a_m)$$

if $\alpha \times_n a_1, \ldots, \alpha \times_n a_m \in W_n$. For $m = 3$ we get standard basis: $\mathbf{e}_1 = \mathbf{i} = (1, 0, 0)$, $\mathbf{e}_2 = \mathbf{j} = (0, 1, 0)$, $\mathbf{e}_3 = \mathbf{k} = (0, 0, 1)$. And for any $\mathbf{a} = (a_1, a_2, a_3) \in E_3 W_n$ we get:

$$\mathbf{a} = a_1 \times_n \mathbf{i} +_n a_2 \times_n \mathbf{j} +_n a_3 \times_n \mathbf{k}$$

Let's consider now what's the difference between classical linear space with dimension m and $E_m W_n$.

We will use the following notations: $\mathbf{a}, \mathbf{b}, \mathbf{c}$ mean vectors, α, β mean scalars.

LS1. Addition commutativity: If $\mathbf{a} +_n \mathbf{b} \in E_m W_n$, then $\mathbf{b} +_n \mathbf{a} \in E_m W_n$ and $\mathbf{a} +_n \mathbf{b} = \mathbf{b} +_n \mathbf{a}$. And inverse, if $\mathbf{b} +_n \mathbf{a} \in E_m W_n$, then $\mathbf{a} +_n \mathbf{b} \in E_m W_n$ and $\mathbf{a} +_n \mathbf{b} = \mathbf{b} +_n \mathbf{a}$. So, it is no difference between classical linear space with dimension m and $E_m W_n$.

LS2. Addition associativity in $E_m W_n$ does not exist. But if $\mathbf{a} +_n \mathbf{b}, (\mathbf{a} +_n \mathbf{b}) +_n \mathbf{c}, \mathbf{b} +_n \mathbf{c}, \mathbf{a} +_n (\mathbf{b} +_n \mathbf{c}) \in E_m W_n$, then

$$(\mathbf{a} +_n \mathbf{b}) +_n \mathbf{c} = \mathbf{a} +_n (\mathbf{b} +_n \mathbf{c})$$

It is important the condition $\mathbf{a} +_n \mathbf{b}, (\mathbf{a} +_n \mathbf{b}) +_n \mathbf{c}, \mathbf{b} +_n \mathbf{c}, \mathbf{a} +_n (\mathbf{b} +_n \mathbf{c}) \in E_m W_n$ we can see from the following example: For $m = 3, \mathbf{a} = (50, 50, 50), \mathbf{b} = (-50, -50, -50), \mathbf{c} = (-50, -50, -50)$, we have $\mathbf{a} +_n \mathbf{b} = \mathbf{0}, (\mathbf{a} +_n \mathbf{b}) +_n \mathbf{c} = \mathbf{c}$, and $\mathbf{b} +_n \mathbf{c}$ does not belong to $E_3 W_n$.

LS3. Zero existance: For all $\mathbf{a}$

$$\mathbf{0} +_n \mathbf{a} = \mathbf{a} +_n \mathbf{0} = \mathbf{a}$$

, where $\mathbf{0} = (0, 0, 0)$. So, it is no difference between classical linear space with dimension m and $E_m W_n$.

LS4. Existence of inverse by addition element: For any $\mathbf{a}$, there is $(-\mathbf{a})$ such that

$$\mathbf{a} +_n (-\mathbf{a}) = \mathbf{0}$$

. So, it is no difference between classical linear space with dimension m and E_mW_n.

LS5. There is no associativity of scalar multiplication. If $\mathbf{a} = (a_1, \ldots, a_m) \in E_mW_n$, $\alpha, \beta \in W_n$, then $\alpha \times_n (\beta \times_n \mathbf{a}) = (\alpha \times_n (\beta \times_n a_1), \ldots, \alpha \times_n (\beta \times_n a_m))$ and $(\alpha \times_n \beta) \times_n \mathbf{a} = ((\alpha \times_n \beta) \times_n a_1, \ldots, (\alpha \times_n \beta) \times_n a_m)$. we have to have $\alpha \times_n (\beta \times_n a_1), \ldots, \alpha \times_n (\beta \times_n a_m), (\alpha \times_n \beta) \times_n a_1, \ldots, (\alpha \times_n \beta) \times_n a_m \in W_n$.

If we take $\alpha = 1, \beta = 1, \mathbf{a} = (1, 1, 1) \in E_3W_n$, then $\alpha \times_n (\beta \times_n \mathbf{a}) = (\alpha \times_n \beta) \times_n \mathbf{a}$. But if we take $\alpha = 0.01, \beta = 0.1, \mathbf{a} = (10, 10, 10) \in E_3W_2$, then $\alpha \times_n (\beta \times_n \mathbf{a}) = (0.01; 0.01; 0.01)$ and $(\alpha \times_n \beta) \times_n \mathbf{a} = (0; 0; 0)$. And we know

$$\delta_3 = \alpha \times_n (\beta \times_n \gamma) -_n (\alpha \times_n \beta) \times_n \gamma, (\alpha, \beta, \gamma \in W_n)$$

is a random variable in W_n, и $\delta_3 = 0$ with probability $P < 1$. So, a probability of the equality

$$\alpha \times_n (\beta \times_n \mathbf{a}) = (\alpha \times_n \beta) \times_n \mathbf{a}$$

is less than 1.

LS6. There is no distributivity of scalar multiplication. If $\mathbf{a} = (a_1, \ldots, a_m) \in E_mW_n, \alpha, \beta \in W_n$, then

$$(\alpha +_n \beta) \times_n \mathbf{a} = ((\alpha +_n \beta) \times_n a_1, \ldots, (\alpha +_n \beta) \times_n a_m)$$

and

$$\alpha \times_n \mathbf{a} +_n \beta \times_n \mathbf{a} = (\alpha \times_n a_1 +_n \beta \times_n a_1, \ldots, \alpha \times_n a_m +_n \beta \times_n a_m)$$

We have to have $(\alpha +_n \beta) \times_n a_1, \ldots, (\alpha +_n \beta) \times_n a_m, \alpha \times_n a_1 +_n \beta \times_n a_1, \ldots, \alpha \times_n a_m +_n \beta \times_n a_m \in W_n$.

If we take $\alpha = 1, \beta = 1, \mathbf{a} = (1, 1, 1) \in E_3W_n$, then $(\alpha +_n \beta) \times_n \mathbf{a} = \alpha \times_n \mathbf{a} +_n \beta \times_n \mathbf{a}$. But if we take $\alpha = 0.03, \beta = 0.07, \mathbf{a} = (0.1, 0.1, 0.1) \in E_3W_2$, then $(\alpha +_n \beta) \times_n \mathbf{a} = (0.01, 0.01, 0.01)$ and $\alpha \times_n \mathbf{a} +_n \beta \times_n \mathbf{a} = (0, 0, 0)$ And we know

$$\delta_2 = (\alpha +_n \beta) \times_n \gamma -_n (\alpha \times_n \gamma +_n \beta \times_n \gamma), \alpha, \beta, \gamma \in W_n$$

is a random variable in W_n, and $\delta_2 = 0$ with probability $P < 1$. So, a probability of the equality

$$(\alpha +_n \beta) \times_n \mathbf{a} = \alpha \times_n \mathbf{a} +_n \beta \times_n \mathbf{a}$$

is less than 1.

LS7. There is no distributivity of scalar multiplication to vector sum. If $\mathbf{a} = (a_1, \ldots, a_m), \mathbf{b} = (b_1, \ldots, b_m) \in E_mW_n, \alpha \in W_n$, then

$$\alpha \times_n (\mathbf{a} +_n \mathbf{b}) = (\alpha \times_n (a_1 +_n b_1), \ldots, \alpha \times_n (a_m +_n b_m))$$

and

$$\alpha \times_n \mathbf{a} +_n \alpha \times_n \mathbf{b} = (\alpha \times_n a_1 +_n \alpha \times_n b_1, \ldots, \alpha \times_n a_m +_n \alpha \times_n b_m)$$

We have to have $\alpha \times_n (a_1 +_n b_1), \ldots, \alpha \times_n (a_m +_n b_m), \alpha \times_n a_1 +_n \alpha \times_n b_1, \ldots, \alpha \times_n a_m +_n \alpha \times_n b_m \in W_n$. If we take $\alpha = 1, \mathbf{a} = (1, 1, 1), \mathbf{b} = (1, 1, 1) \in E_3W_n$, then $\alpha \times_n (\mathbf{a} +_n \mathbf{b}) = \alpha \times_n \mathbf{a} +_n \alpha \times_n \mathbf{b}$.

But if we take $\alpha = 0.01, \mathbf{a} = (0.6, 0.6, 0.6), \mathbf{b} = (0.4, 0.4, 0.4) \in E_3W_2$, then $\alpha \times_n (\mathbf{a} +_n \mathbf{b}) = (0.01, 0.01, 0.01)$ and $\alpha \times_n \mathbf{a} +_n \alpha \times_n \mathbf{b} = (0, 0, 0)$.

So, a probability of the equality

$$\alpha \times_n (\mathbf{a} +_n \mathbf{b}) = \alpha \times_n \mathbf{a} +_n \alpha \times_n \mathbf{b}$$

is less than 1.

LS8. Scalar multiplication unit:

$$1 \times_n \mathbf{a} = \mathbf{a}$$

. So, it is no difference between classical linear space with dimension m and E_mW_n.

4.2 Scalar product in E_mW_n

We name a scalar product of vectors $\mathbf{a} = (a_1, \dots, a_m), \mathbf{b} = (b_1, \dots, b_m) \in E_mW_n$ the following sum:

$$(\mathbf{a}, \mathbf{b}) = (\dots (a_1 \times_n b_1 +_n \dots) +_n a_m \times_n b_m$$

Let's consider now what's the difference between classical scalar product in linear space with dimension m and scalar product in E_mW_n.

SP1. Scalar product in E_mW_n is commutative:

$$(\mathbf{a}, \mathbf{b}) = (\mathbf{b}, \mathbf{a})$$

. So, it is no difference between classical linear space with dimension m and E_mW_n.

SP2. Scalar product in E_mW_n is not distributive. If

$$\mathbf{a} = (a_1, \dots, a_m), \mathbf{b} = (b_1, \dots, b_m), \mathbf{c} =$$

$$= (c_1, \dots, c_m) \in E_mW_n$$

, then

$$(\mathbf{a}, (\mathbf{b} +_n \mathbf{c})) = (\dots (a_1 \times_n (b_1 +_n c_1) +_n \dots) +_n a_m \times_n (b_m +_n c_m)$$

$$(\mathbf{a}, \mathbf{b}) +_n (\mathbf{a}, \mathbf{c}) = (a_1 \times_n b_1 +_n a_1 \times_n c_1) +_n \dots) +_n (a_m \times_n b_m +_n a_m \times_n c_m)$$

We have to assume that all elements of these equalities are in W_n. If we take $\mathbf{a} = (1, 1, 1), \mathbf{b} = (1, 1, 1), \mathbf{c} = (1, 1, 1) \in E_3W_n$, then $(\mathbf{a}, (\mathbf{b} +_n \mathbf{c})) = (\mathbf{a}, \mathbf{b}) +_n (\mathbf{a}, \mathbf{c})$. But if we take $\mathbf{a} = (0.01, 0.01, 0.01), \mathbf{b} = (0.6, 0.6, 0.6), \mathbf{c} = (0.4, 0.4, 0.4) \in E_3W_2$, then $(\mathbf{a}, (\mathbf{b} +_n \mathbf{c})) = 0.03$ and $(\mathbf{a}, \mathbf{b}) +_n (\mathbf{a}, \mathbf{c}) = 0$.

So, a probability of

$$(\mathbf{a}, (\mathbf{b} +_n \mathbf{c})) = (\mathbf{a}, \mathbf{b}) +_n (\mathbf{a}, \mathbf{c})$$

is less than 1.

SP3. Scalar multiplication on scalar product in E_mW_n is not associative. If

$$\mathbf{a} = (a_1, \dots, a_m)$$

,

$$\mathbf{b} = (b_1, \ldots, b_m) \in E_m W_n, \alpha \in W_n$$

, then

$$\alpha \times_n (\mathbf{a}, \mathbf{b}) =$$

$$= \alpha \times_n (\ldots (a_1 \times_n b_1 +_n \ldots) +_n a_m \times_n b_m)$$

$$((\alpha \times_n \mathbf{a}), \mathbf{b}) =$$

$$= (\ldots (\alpha \times_n a_1) \times_n b_1 +_n \ldots) +_n (\alpha \times_n a_m) \times_n b_m$$

We have to assume that all elements of these equalities are in W_n. If we take $\alpha = 1, \mathbf{a} = (1,1,1), \mathbf{b} = (1,1,1) \in E_3 W_n$, then $\alpha \times_n (\mathbf{a}, \mathbf{b}) = ((\alpha \times_n \mathbf{a}), \mathbf{b})$. But if we take $\alpha = 0.01, \mathbf{a} = (0.1, 0.3, 0.6), \mathbf{b} = (1,1,1) \in E_3 W_2$, then $\alpha \times_n (\mathbf{a}, \mathbf{b}) = 0.01$ and $((\alpha \times_n \mathbf{a}), \mathbf{b}) = 0$.

So, a probability of

$$\alpha \times_n (\mathbf{a}, \mathbf{b}) = ((\alpha \times_n \mathbf{a}), \mathbf{b})$$

is less than 1.

SP4. Vector's length definition: Square of vector's length is $|\mathbf{a}|^2 = (\ldots (a_1 \times_n a_1 +_n \ldots) +_n a_m \times_n a_m$. Square of length $|a|^2$ of vector $\mathbf{a} = (a_1, \ldots, a_m)$ always exists (if $(\ldots (a_1 \times_n a_1 +_n \ldots) +_n a_m \times_n a_m \in W_n)$, but length itself calculated as $\sqrt{|\mathbf{a}|^2} \in W_n$ exists not always (see **?**). For example, vectors of standard basis $\mathbf{i} = (1,0,0), \mathbf{j} = (0,1,0), \mathbf{k} = (0,0,1)$ have length 1 for any W_n. But vector $\mathbf{a} = (0.7, 0.1, 0) \in E_3 W_2$ doesn't have a length. Generally definition in **SP4** works for $E_m W_n$ with some probability less than 1.

SP5. Vector's perpendicularity definition: Vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, if $(\mathbf{a}, \mathbf{b}) = 0$.

SP6. Vector's parallelism definition: Vectors $\mathbf{a}, \mathbf{b}$ are parallel ($\mathbf{a} || \mathbf{b}$) if there are exist $\alpha \in W_n$, or $\beta \in W_n$ such that $\mathbf{b} = \alpha \times_n \mathbf{a}$ or $\mathbf{a} = \beta \times_n \mathbf{b}$. Two non-zero vectors in $E_m W_n$ may be perpendicular and parallel same time. For example, if $\mathbf{a} = (0.02, 0.04, 0.01) \in E_3 W_2, \alpha = 2, \mathbf{b} = \alpha \times_n \mathbf{a} = (0.04, 0.08, 0.02)$, то $\mathbf{a} || \mathbf{b}$ and $(\mathbf{a}, \mathbf{b}) = 0$. We can see also:

$$(\mathbf{i}, \mathbf{i}) = (\mathbf{j}, \mathbf{j}) = (\mathbf{k}, \mathbf{k}) = 1$$

$$|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$$

$$(\mathbf{i}, \mathbf{j}) = (\mathbf{i}, \mathbf{k}) = (\mathbf{k}, \mathbf{j}) = 0$$

i.e. $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is ortonormal basis in $E_3 W_n$.

We have the following statements for classical linear algebra:

T1. Each vector in m- dimensional linear space is a linear combination of m basis vectors;

T2. For any new basis with vectors $\mathbf{e'}_i$ and any old basis with vectors $\mathbf{e}_i$ transition matrix always exists;

T3. Each vector existing in old basis $\mathbf{e}_i$, always exists in new basis $\mathbf{e'}_i$;

T4. In Eucledian space for each vector there is parallel unit vector.

Let's consider now Observer's Mathematics point of view.

Let's start from **T1**. Let's consider E_3W_n with basis $\mathbf{e}_1 = (3,0,0), \mathbf{e}_2 = (0,3,0), \mathbf{e}_3 = (0,0,3)$, take vector $\mathbf{a} = (1,1,1)$. We know (see **?**), there are not $\alpha, \beta, \gamma \in W_n$ such that $\alpha \times_n \mathbf{e}_1 +_n \beta \times_n \mathbf{e}_2 +_n \gamma \times_n \mathbf{e}_3 = \mathbf{a}$. We have same situation for any E_mW_n with $m > 3$. Sure, if we take $\mathbf{e}_1 = \mathbf{i}, \mathbf{e}_2 = \mathbf{j}, \mathbf{e}_3 = \mathbf{k}$, then each vector in E_3W_n is a linear combination of basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$. So, statement **T1** is correct in E_3W_n with some probability less than 1.

Let's consider now **T2**. If we take old basis in E_3W_n $\mathbf{e}_1 = (3,0,0), \mathbf{e}_2 = (0,3,0), \mathbf{e}_3 = (0,0,3)$ and new basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$, transition matrix doesn't exist (see **?**). But if we take old basis in E_3W_n $\mathbf{e}_1 = (2,0,0), \mathbf{e}_2 = (0,2,0), \mathbf{e}_3 = (0,0,2)$ and new basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$, transition matrix A exists:

$$A = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$

So, statement **T2** is correct in E_3W_n with some probability less than 1.

Let's consider now **T3**. We consider now E_3W_n from point of view $W_m, m = 6n$. Basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$ defines a lattice in E_3W_n: the set of vectors (points) $(a_1, a_2, a_3), a_1, a_2, a_3 \in W_n$. If we take old basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$ and new basis **i'**, **j'**, **k'**, where **i'** is unit vector on direction of line $y = x$ (in first quadrant), **j'** is unit vector on direction of line $y = -x$ (in second quadrant), **k'** $= \mathbf{k}$. Basis **i'**, **j'**, **k'** also defines it's lattice $(b_1, b_2, b_3), b_1, b_2, b_3 \in W_n$. And we can see that these two lattices don't intersect $n = 2, a_1, a_2, a_3, b_1, b_2, b_3 \in (0, 0.89)$. So, if we take vector $\mathbf{a} = (a_1, a_2, 0)$ in first lattice, this vector doesn't exist in second lattice, and inverse, if we take vector $\mathbf{b} = (b_1, b_2, 0)$ in second lattice, this vector doesn't exist in first lattice. So, statement **T3** is correct in E_3W_n with some probability less than 1.

Finally, let's consider **T4**. If we take vector $\mathbf{a} = (a_1, a_2, a_3) \in E_3W_n$ with $a_1 = a_2 = a_3 = 1$, then $\alpha \times_n \mathbf{a} = (\alpha, \alpha, \alpha)$. For $n = 2$ and

$$\alpha \in \pm 0.50, \pm 0.51, \pm 0.52, \pm 0.53, \pm 0.54, \pm 0.55, \pm 0.56, \pm 0.57, \pm 0.58, \pm 0.59$$

we have

$$\alpha^2 +_n \alpha^2 +_n \alpha^2 = 0.75 < 1$$

For

$$\alpha \in \pm 0.60, \pm 0.61, \pm 0.62, \pm 0.63, \pm 0.64, \pm 0.65, \pm 0.66, \pm 0.67, \pm 0.68, \pm 0.69$$

we have

$$\alpha^2 +_n \alpha^2 +_n \alpha^2 = 1.08 > 1$$

That means that such α with $|\alpha \times_n \mathbf{a}| = 1$ doesn't exist. From another side, vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ have a length 1. So, statement **T4** is correct in E_3W_n with some probability less than 1.

Tensors are geometric objects with linear relations between them. Elementary examples of such relations include the scalar product, the cross product, and linear maps. Euclidean vectors and scalars themselves are also tensors. Just as the components of a vector change when we change the basis of the vector space, the components of a tensor also change under such a transformation. Each tensor comes equipped with a transformation law that details how the components of the tensor respond to a change of basis. The components of a vector can respond in two distinct ways to a change of basis (covariance and contravariance of vectors), where the new basis vectors $\mathbf{e}'_i$, $i = 1, \ldots, m$, are expressed in terms of the old basis vectors $\mathbf{e}_j$, $j = 1, \ldots, m$ as

$$\mathbf{e}'_i = R^1_i \mathbf{e}_1 + \ldots + R^m_i \mathbf{e}_m$$

Here R^j_i are the entries of the change of basis matrix. The components v^i of a column vector $\mathbf{v}$ transform with the inverse of the matrix R,

$$v'_i = {R^{-1}}^i_1 v^1 + \ldots + {R^{-1}}^i_m v^m$$

where the v'^i are the components in the new basis. In contrast, the components, w_i of a covector (or row vector), w transform with the matrix R itself,

$$w'_i = R^1_i w_1 + \ldots + R^m_i w_m$$

The components of a more general tensor transform by some combination of covariant and contravariant transformations, with one transformation law for each index. The transformation law for an order $p+q$ tensor with p contravariant indices and q covariant indices is thus given as,

$$\hat{T}^{i'_1,\ldots,i'_p}_{j'_1,\ldots,j'_q} = \left(R^{-1}\right)^{i'_1}_{i_1} \ldots \left(R^{-1}\right)^{i'_p}_{i_p} R^{j_1}_{j'_1} \ldots R^{j_q}_{j'_q} T^{i_1,\ldots,i_p}_{j_1,\ldots,j_q}$$

Here the primed indices denote tensor components in the new coordinates, and the unprimed indices denote the tensor components in the old coordinates. Such a tensor called as a (p,q)-tensor.

And we get

T5. Tensors of type (p,q) in classical Linear Algebra are not tensors in Observer's Mathematics. They are only tensors with some probability less than 1.

4.3 Vector product in E_3W_n

We name a vector product of vectors$\mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3) \in E_3W_n$ the following:

$$\mathbf{a} \times \mathbf{b} = (a_2 \times_n b_3 -_n a_3 \times_n b_2) \times_n \mathbf{i} -_n (a_1 \times_n b_3 -_n a_3 \times_n b_1) \times_n \mathbf{j} +_n (a_1 \times_n b_2 -_n a_2 \times_n b_1) \times_n \mathbf{k}$$

Let's consider now what's the difference between classical vector product in linear space with dimension 3 and vector product in E_3W_n.

We have: For $\mathbf{a} = \mathbf{i} = (1,0,0), \mathbf{b} = \mathbf{j} = (0,1,0), \mathbf{c} = \mathbf{k} = (0,0,1)$ we have in E_3W_n same connections as in classical linear algebra:

$$\mathbf{i} = \mathbf{j} \times \mathbf{k}, \mathbf{j} = \mathbf{k} \times \mathbf{i}, \mathbf{k} = \mathbf{i} \times \mathbf{j}$$

$$\mathbf{k} \times \mathbf{j} = -\mathbf{i}, \mathbf{i} \times \mathbf{k} = -\mathbf{j}, \mathbf{j} \times \mathbf{i} = -\mathbf{k}$$

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{k} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$$

Also, we have same relations in E_3W_n as in classical linear algebra:

CP1.

$$\mathbf{a} \times \mathbf{a} = \mathbf{0}$$

for any $\mathbf{a} \in E_3W_n$.

CP2.

$$\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$$

for any $\mathbf{a}, \mathbf{b} \in E_3W_n$.

But the following statements, in E_3W_n are correct with some probability less than 1:

CP3. Vector product in E_3W_n is not distributive.

For any $\mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3), \mathbf{c} = (c_1, c_2, c_3) \in E_3W_n$

$$\mathbf{a} \times (\mathbf{b} +_n \mathbf{c}) = ((a_2 \times_n (b_3 +_n c_3) -_n a_3 \times_n (b_2 +_n c_2)) \times_n \mathbf{i} -_n$$
$$-_n (a_1 \times_n (b_3 +_n c_3) -_n (a_3 \times_n (b_1 +_n c_1)) \times_n \mathbf{j} +_n$$
$$+_n (a_1 \times_n (b_2 +_n c_2) -_n a_2 \times_n (b_1 +_n c_1)) \times_n \mathbf{k}$$

$$\mathbf{a} \times \mathbf{b} = (a_2 \times_n b_3 -_n a_3 \times_n b_2) \times_n \mathbf{i} -_n$$
$$-_n (a_1 \times_n b_3 -_n a_3 \times_n b_1) \times_n \mathbf{j} +_n$$
$$+_n (a_1 \times_n b_2 -_n a_2 \times_n b_1) \times_n \mathbf{k}$$

$$\mathbf{a} \times \mathbf{c} = (a_2 \times_n c_3 -_n a_3 \times_n c_2) \times_n \mathbf{i} -_n$$
$$-_n (a_1 \times_n c_3 -_n a_3 \times_n c_1) \times_n \mathbf{j} +_n$$
$$+_n (a_1 \times_n c_2 -_n a_2 \times_n c_1) \times_n \mathbf{k}$$

$$\mathbf{a} \times \mathbf{b} +_n \mathbf{a} \times \mathbf{c} = ((a_2 \times_n b_3 -_n a_3 \times_n b_2) +_n ((a2 \times_n c_3 -_n a_3 \times_n c_2)) \times_n \mathbf{i} -_n$$
$$-_n ((a_1 \times_n b_3 -_n a_3 \times_n b_1) +_n (a_1 \times_n c_3 -_n a_3 \times_n c_1)) \times_n \mathbf{j} +_n$$
$$+_n ((a_1 \times_n b_2 -_n a_2 \times_n b_1) +_n (a_1 \times_n c_2 -_n a_2 \times_n c_1)) \times_n \mathbf{k}$$

All elements of these equalities have to be in W_n. If we take

$$\mathbf{a} = (1, 1, 1), \mathbf{b} = (1, 1, 1), \mathbf{c} = (1, 1, 1) \in E_3W_n$$

, then

$$\mathbf{a} \times (\mathbf{b} +_n \mathbf{c}) =$$
$$= \mathbf{a} \times \mathbf{b} +_n \mathbf{a} \times \mathbf{c}$$

. But if we take

$$\mathbf{a} = (0.01, 0.02, 0.03), \mathbf{b} = (0.6, 0.6, 0.6)$$

,

$$\mathbf{c} = (0.4, 0.4, 0.4) \in E_3W_2$$

, then

$$\mathbf{a} \times (\mathbf{b} +_n \mathbf{c}) = (-0.01, 0.02, -0.01)$$

and

$$\mathbf{a} \times \mathbf{b} +_n \mathbf{a} \times \mathbf{c} = (0, 0, 0)$$

.

So, a probability of equality

$$\mathbf{a} \times (\mathbf{b} +_n \mathbf{c}) = \mathbf{a} \times \mathbf{b} +_n \mathbf{a} \times \mathbf{c}$$

is less than 1.

CP4. Scalar multiplication on vector product in E_3W_n is not associative. For any $\mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3) \in E_3W_n$ and for any scalar $\alpha \in W_n$

$$(\alpha \times_n \mathbf{a}) \times \mathbf{b} = ((\alpha \times_n a_2) \times_n b_3 -_n (\alpha \times_n a_3) \times_n b_2) \times_n \mathbf{i} -_n$$
$$-_n((\alpha \times_n a_1) \times_n b_3 -_n (\alpha \times_n a_3) \times_n b_1) \times_n \mathbf{j} +_n$$
$$+_n((\alpha \times_n a_1) \times_n b_2 -_n (\alpha \times_n a_2) \times_n b_1) \times_n \mathbf{k}$$

$$\mathbf{a} \times (\alpha \times_n \mathbf{b}) = ((a_2 \times_n (\alpha \times_n b_3) -_n a_3 \times_n (\alpha \times_n b_2)) \times_n \mathbf{i} -_n$$
$$-_n(a_1 \times_n (\alpha \times_n b_3) -_n a_3 \times_n (\alpha \times_n b_1)) \times_n \mathbf{j} +_n$$
$$+_n(a_1 \times_n (\alpha \times_n b_2) -_n a_2 \times_n (\alpha \times_n b_1)) \times_n \mathbf{k}$$

$$\alpha \times_n (\mathbf{a} \times \mathbf{b}) = (\alpha \times_n (a_2 \times_n b_3 -_n a_3 \times_n b_2)) \times_n \mathbf{i} -_n$$
$$-_n(\alpha \times_n (a_1 \times_n b_3 -_n a_3 \times_n b_1)) \times_n \mathbf{j} +_n$$
$$+_n(\alpha \times_n (a_1 \times_n b_2 -_n a_2 \times_n b_1)) \times_n \mathbf{k}$$

All elements of these equalities have to be in W_n. If we take $\alpha = 1, \mathbf{a} = (1, 1, 1), \mathbf{b} = (1; 1; 1) \in E_3W_n$, then $\alpha \times_n (\mathbf{a} \times \mathbf{b}) = (\alpha \times_n \mathbf{a}) \times \mathbf{b}$ But if we take $\alpha = 0.01, \mathbf{a} = (0.1, 0.3, 0.6), \mathbf{b} = (10; 10; 10) \in E_3W_2$, then $\alpha \times_n (\mathbf{a} \times \mathbf{b}) = (-0.03, 0, 05, -0.02)$ and $(\alpha \times_n \mathbf{a}) \times \mathbf{b} = (0, 0, 0)$.

So, a probability of equality

$$\alpha \times_n (\mathbf{a} \times \mathbf{b}) = (\alpha \times_n \mathbf{a}) \times \mathbf{b}$$

is less than 1.

CP5.

Equality

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}, \mathbf{c}) \times_n \mathbf{b} -_n (\mathbf{a}, \mathbf{b}) \times_n \mathbf{c}$$

is correct in E_3W_n with probability less than 1.

For any $\mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3), \mathbf{c} = (c_1, c_2, c_3) \in E_3W_n$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \times ((b_2 \times_n c_3 -_n b_3 \times_n c_2) \times_n \mathbf{i} -_n$$

$$-_n(b_1 \times_n c_3 -_n b_3 \times_n c_1) \times_n \mathbf{j} +_n$$

$$+_n(b_1 \times_n c_2 -_n b_2 \times_n c_1) \times_n \mathbf{k} =$$

$$= (a_2 \times_n (b_1 \times_n c_2 -_n b_2 \times_n c_1) -_n a_3 \times_n (-b_1 \times_n c_3 +_n b_3 \times_n c_1)) \times_n \mathbf{i} -_n$$

$$-_n(a_1 \times_n (b_1 \times_n c_2 -_n b_2 \times_n c_1) -_n a_3 \times_n (b_2 \times_n c_3 -_n b_3 \times_n c_2)) \times_n \mathbf{j} +_n$$

$$+_n(a_1 \times_n (-b_1 \times_n c_3 +_n b_3 \times_n c_1) -_n a_2 \times_n (b_1 \times_n c_2 -_n b_2 \times_n c_1)) \times_n \mathbf{k}$$

$$(\mathbf{a}, \mathbf{c}) \times_n \mathbf{b} = ((a_1 \times_n c_1 +_n a_2 \times_n c_2) +_n a_3 \times_n c_3) \times_n b_1) \times_n \mathbf{i} +_n$$

$$+_n((a_1 \times_n c_1 +_n a_2 \times_n c_2) +_n a_3 \times_n c_3) \times_n b_2) \times_n \mathbf{j} +_n$$

$$+_n((a_1 \times_n c_1 +_n a_2 \times_n c_2) +_n a_3 \times_n c_3) \times_n b_3) \times_n \mathbf{k}$$

$$(\mathbf{a}, \mathbf{b}) \times_n \mathbf{c} = ((a_1 \times_n b_1 +_n a_2 \times_n b_2) +_n a_3 \times_n b_3) \times_n c_1) \times_n \mathbf{i} +_n$$

$$+_n((a_1 \times_n b_1 +_n a_2 \times_n b_2) +_n a_3 \times_n b_3) \times_n c_2) \times_n \mathbf{j} +_n$$

$$+_n((a_1 \times_n b_1 +_n a_2 \times_n b_2) +_n a_3 \times_n b_3) \times_n c_3) \times_n \mathbf{k}$$

All elements of these equalities have to be in W_n. If we take $\mathbf{a} = (1, 1, 1), \mathbf{b} = (1; 1; 1), \mathbf{c} = (1; 1; 1) \in E_3W_n$, then

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}, \mathbf{c}) \times_n \mathbf{b} -_n (\mathbf{a}, \mathbf{b}) \times_n \mathbf{c}$$

But if we take $\mathbf{a} = (0.01, 0.01, 0.01), \mathbf{b} = (0.6; 0.6; 0.6), \mathbf{c} = (0.4; 0.9; -0.9) \in E_3W_2$, then $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (0; -0.01; 0.01)$ and $(\mathbf{a}, \mathbf{c}) \times_n \mathbf{b} -_n (\mathbf{a}, \mathbf{b}) \times_n \mathbf{c} = (0, 0, 0)$

So, a probability of equality

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}, \mathbf{c}) \times_n \mathbf{b} -_n (\mathbf{a}, \mathbf{b}) \times_n \mathbf{c}$$

is less than 1.

CP6. Vector product, vector's parallelism and perpendicularity.

We named vectors $\mathbf{a}, \mathbf{b}$ are parallel $(\mathbf{a}||\mathbf{b})$, if there are $\alpha, \beta \in W_n$ such that $\mathbf{b} = \alpha \times_n \mathbf{a}$ or $\mathbf{a} = \beta \times_n \mathbf{b}$. For any $\alpha \in W_n, \mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = \alpha \times_n \mathbf{a} = (\alpha \times_n a_1, \alpha \times_n a_2, \alpha \times_n a_3) \in E_3W_n$

$$\mathbf{a} \times \mathbf{b} = ((a_2 \times_n (\alpha \times_n a_3) -_n a_3 \times_n (\alpha \times_n a_2)) \times_n \mathbf{i} -_n$$

$$-_n(a_1 \times_n (\alpha \times_n a_3) -_n a_3 \times_n (\alpha \times_n a_1)) \times_n \mathbf{j} +_n$$
$$+_n(a_1 \times_n (\alpha \times_n a_2) -_n a_2 \times_n (\alpha \times_n a_1)) \times_n \mathbf{k}$$

And $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ also with probability less than 1, because we know that

$$\delta_2 = (\alpha +_n \beta) \times_n \gamma -_n (\alpha \times_n \gamma +_n \beta \times_n \gamma), \alpha, \beta, \gamma \in W_n$$

$$\delta_3 = \alpha \times_n (\beta \times_n \gamma) -_n (\alpha \times_n \beta) \times_n \gamma, (\alpha, \beta, \gamma \in W_n)$$

are the random variables in W_n, and $\delta_2 = \delta_3 = 0$ with probability $P < 1$. But difference between $\mathbf{a} \times \mathbf{b}$ and $\mathbf{0}$ is decreasing with growing n. Below we give some example. Let's $\mathbf{b} = \alpha \times_n \mathbf{a} = (\alpha \times_n a_1, \alpha \times_n a_2, \alpha \times_n a_3)$, with $\alpha, \alpha \times_n a_1, \alpha \times_n a_2, \alpha \times_n a_3 \in W_n$

Let's take

$$n = 4, \mathbf{a} = (3.1549, 2.9807, 1.7362), \alpha = 4.4697$$

Then

$$\mathbf{b} = (14.1002, 13.3211, 7.759)$$

$$\mathbf{a} \times \mathbf{b} = -0.0018\mathbf{i} +_n 0.0031\mathbf{j} -_n 0.0019\mathbf{k}$$

$$(\mathbf{a}, \mathbf{a} \times \mathbf{b}) = -0.0055 +_n 0.0089 -_n 0.014 = -0.0106$$

$$(\mathbf{b}, \mathbf{a} \times \mathbf{b}) = -0.0253 +_n 0.0412 -_n 0.014 = 0.0019$$

Let's take

$$n = 6, \mathbf{a} = (3.154932, 2.980749, 1.736284), \alpha = 4.469731$$

Then

$$\mathbf{b} = (14.101681, 13.323129, 7.760707)$$

$$\mathbf{a} \times \mathbf{b} = -0.000021\mathbf{i} +_n 0.000020\mathbf{j} -_n 0.000003\mathbf{k}$$

$$(\mathbf{a}, \mathbf{a} \times \mathbf{b}) = -0.000065 +_n 0.000058 -_n 0.000003 = -0.00001$$

$$(\mathbf{b}, \mathbf{a} \times \mathbf{b}) = -0.000296 +_n 0.000266 -_n 0.000021 = -0.000051$$

Let's take

$$n = 8, \mathbf{a} = (3.15493269, 2.98074951, 1.73628439), \alpha = 4.46973129$$

Then

$$\mathbf{b} = (14.10170118, 13.32314907, 7.76072442)$$

$$\mathbf{a} \times \mathbf{b} = -0.00000021\mathbf{i} +_n 0.00000051\mathbf{j} -_n 0.00000036\mathbf{k}$$

$$(\mathbf{a}, \mathbf{a} \times \mathbf{b}) = -0.00000065 +_n 0.00000147 -_n 0.00000057 = 0.00000025$$

$$(\mathbf{b}, \mathbf{a} \times \mathbf{b}) = -0.00000296 +_n 0.00000678 -_n 0.00000273 = 0.00000109$$

If we take $\mathbf{a} = \mathbf{i}, \mathbf{b} = \mathbf{j}$, then $\mathbf{a}$ and $\mathbf{b}$ are not parallel, and $\mathbf{a} \times \mathbf{b} \neq \mathbf{0}$. But if we take $\mathbf{a} = (0.01, 0.05, 0.07), \mathbf{b} = (0.07, 0.02, 0.06), \mathbf{a}, \mathbf{b} \in E_3W_2$ then $\mathbf{a}$ and $\mathbf{b}$ are not parallel, and $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

For any $\mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3) \in E_3W_n$

$$(\mathbf{a}, \mathbf{a} \times \mathbf{b}) = (a_1 \times_n (a_2 \times_n b_3 -_n a_3 \times_n b_2) -_n a_2 \times_n (a_1 \times_n b_3 -_n a_3 \times_n b_1)) +_n a_3 \times_n (a_1 \times_n b_2 -_n a_2 \times_n b_1)$$

And $(\mathbf{a}, \mathbf{a} \times \mathbf{b}) = \mathbf{0}$ also with probability less than 1. So, the probability of correctness of the following statements: "If $\mathbf{a}$ and $\mathbf{b}$ are parallel ($\mathbf{a}||\mathbf{b}$), then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$" and "If $\mathbf{a}$ and $\mathbf{b}$ are not parallel, then $\mathbf{a} \times \mathbf{b} \neq 0$ and $(\mathbf{a}, \mathbf{a} \times \mathbf{b}) = (\mathbf{b}, \mathbf{a} \times \mathbf{b}) = 0$" is less than 1.

CP7. Equality $(\mathbf{a}, \mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}, \mathbf{c})$ is correct in E_3W_n with probability less than 1.

For any $\mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3), \mathbf{c} = (c_1, c_2, c_3) \in E_3W_n$

$$(\mathbf{a}, \mathbf{b} \times \mathbf{c}) =$$
$$= (a_1 \times_n (b_2 \times_n c_3 -_n b_3 \times_n c_2) -_n a_2 \times_n (b_1 \times_n c_3 -_n b_3 \times_n c_1)) +_n$$
$$+_n a_3 \times_n (b_1 \times_n c_2 -_n b_2 \times_n c_1)$$
$$(\mathbf{a} \times \mathbf{b}, \mathbf{c}) =$$
$$= ((a_2 \times_n b_3 -_n a_3 \times_n b_2) \times_n c_1 -_n (a_1 \times_n b_3 -_n a_3 \times_n b_1) \times_n c_2) +_n$$
$$+_n (a_1 \times_n b_2 -_n a_2 \times_n b_1) \times_n c_3$$

All elements of these equalities have to be in W_n. If we take

$$\mathbf{a} = (1, 1, 1), \mathbf{b} = (1; 1; 1)$$

,

$$\mathbf{c} = (1; 1; 1) \in E_3W_n$$

, then

$$(\mathbf{a}, \mathbf{b} \times \mathbf{c}) =$$
$$= (\mathbf{a} \times \mathbf{b}, \mathbf{c})$$

. But if we take

$$\mathbf{a} = (0.01, 0.01, 0.01), \mathbf{b} = (0.6; 0.6; 0.6)$$

,

$$\mathbf{c} = (0.4; 0.9; -0.9) \in E_3W_2$$

, then

$$(\mathbf{a}, \mathbf{b} \times \mathbf{c}) = -0.01$$

and

$$(\mathbf{a} \times \mathbf{b}, \mathbf{c}) = 0$$

. And

$$(\mathbf{a}, \mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}, \mathbf{c})$$

with probability less than 1.