6. LAGRANGIAN

Let's consider the Lagrangian for a free particle in classical mechanics. Consider the simplest case, that of the free motion of a particle relative to an inertial frame of reference. The Lagrangian in this case can depend only on the square of the velocity. To discover the form of this dependence, we make use of Galileo's relativity principle. If an inertial frame K is moving with an infinitesimal velocity ε relative to another inertial frame K', then $\mathbf{v}' = \mathbf{v} + \varepsilon$. Since the equations of motion must have the same form in every frame, the Lagrangian $L(v^2)$ must be converted by this transformation into a function L' which differs from $L(v^2)$, if at all, only by the total time derivative of a function of coordinates and time.

We have $L' = L(v'^2) = L\left(v^2 + 2\mathbf{v} \cdot \varepsilon + \varepsilon^2\right)$. Expanding this expression in powers of ε and neglecting terms above the first order, we obtain

$$L(v'^2) = L(v^2) + \frac{\partial L}{\partial v^2} 2\mathbf{v} \cdot \varepsilon$$

The second term on the right of this equation is a total time derivative only if it a linear function of the velocity $\mathbf{v}$. Hence $\frac{\partial L}{\partial v^2}$ is independent of the velocity, i.e., the Lagrangian is in this case proportional to the square of the velocity, and we write it as

$$L = \frac{1}{2} m v^2$$

From the fact that a Lagrangian of this form satisfies Galileo's relativity principle for an infinitesimal relative velocity, it follows at once that the Lagrangian is invariant for a finite relative velocity $\mathbf{V}$ of the frames K and K'. For

$$L' = \frac{1}{2} m v'^2 = \frac{1}{2} m \left(\mathbf{v} + \mathbf{V}\right)^2 = \frac{1}{2} m v^2 + m\mathbf{v} \cdot \mathbf{V} + \frac{1}{2} m \mathbf{V}^2$$

or

$$L' = L + \frac{d\left(m\mathbf{v} \cdot \mathbf{V} + \frac{1}{2}\mathbf{V}^2 t\right)}{dt}$$

The second term is a total time derivative and may be omitted.

Let's consider the Lagrangian for a free particle in special relativity. The principle of Least Action states that a mechanical system should have a quantity called the action S. Such quantity is minimized (in other words, $\delta S = 0$ for the actual motion of the system. The action of a relativistic system should be

1. a scalar, that means Lorentz transformations will not affect this quantity,
2. an integral of which the integrand is a first-order differential.

The only quantity that satisfies the two criteria above is the space-time interval ds, or a scalar multiple thereof. In short, we can conclude that the action must have the following form: $S = \kappa \int ds$. We have

$$ds = \sqrt{c^2 dt - dx^2 - dy^2 - dz^2}$$

After pulling out cdt from the square root and noting that $\frac{dx^2+dy^2+dz^2}{dt^2} = \upsilon^2$, we have $c^2dt^2 - dx^2 - dy^2 - dz^2 = c^2dt^2 - \upsilon^2dt^2 = (c^2 - \upsilon^2)\,dt$ and thus

$$ds = cdt\sqrt{1 - \frac{\upsilon^2}{c^2}}$$

Hence

$$S = c\kappa \int \sqrt{1 - \frac{\upsilon^2}{c^2}}dt$$

Now, the action integral can be expressed as a time integral of the Lagrangian between two fixed times:

$$S = \int Ldt$$

Then we can just read off the Lagrangian:

$$L = c\kappa\sqrt{1 - \frac{\upsilon^2}{c^2}}$$

What is remaining now is determining the expression for κ. At this point we should note that for low velocity υ, this relativistic expression for the Lagrangian should resemble that of the classical free Lagrangian $L = \frac{1}{2}m\upsilon^2$. To compare the two Lagrangians, we perform a Taylor expansion on the square root:

$$L = c\kappa\left(1 - \frac{\upsilon^2}{2c^2} + O(\upsilon^4)\right)$$

The firs term, $c\kappa$, is a constant. That will not affect the equations of motion (for example, Euler-Lagrange Equation). The second term, after expanding out, is equal to $-\kappa\frac{\upsilon^2}{2c}$. To reduce to the classical limit, we can put $\kappa = -mc$. Therefore, the relativistic Lagrangian is:

$$L = -mc^2\sqrt{1 - \frac{\upsilon^2}{c^2}}$$

Let us consider the Observer's Mathematics point of view.

Theorem 6.1. *In classical mechanics,* $P\left(L = \frac{m\upsilon^2}{2}\right) < 1$*, where* P *is the probability.*

Theorem 6.2. *In special relativity,* $P\left(L = -mc^2\sqrt{1 - \frac{\upsilon^2}{c^2}}\right) < 1$*, where* P *is the probability.*